arXiv:hep-th/9410185v2 31 Oct 1994

PHASE SPACE PROPERTIES OF CHARGED FIELDS IN THEORIES OF LOCAL OBSERVABLES

Dedicated to B. Schroer

D. BUCHHOLZ[a)] and C. D'ANTONI[b)]*

[a)] II. Institut für Theoretische Physik,
Universität Hamburg D–22761 Hamburg, Germany.
[b)]Dipartimento di Matematica, Università di Roma
"Tor Vergata" I–00133 Rome, Italy.

ABSTRACT. Within the setting of algebraic quantum field theory a relation between phase-space properties of observables and charged fields is established. These properties are expressed in terms of compactness and nuclearity conditions which are the basis for the characterization of theories with physically reasonable causal and thermal features. Relevant concepts and results of phase space analysis in algebraic quantum field theory are reviewed and the underlying ideas are outlined.

1. Introduction

In the general structural analysis of relativistic quantum field theories it has proved to be useful to characterize the phase space properties of a theory with the help of compactness or nuclearity conditions. These conditions have been a key to the understanding of several physically significant issues, such as the problem of causal (statistical) independence [1-5] and the existence and structure of thermal equilibrium states [6-8]. They are also an important ingredient in the analysis of the particle aspects of the theory [9-12].

The heuristic basis of these conditions is the following physical idea, which is almost as old as quantum theory itself [13]: In any physically reasonable theory

---

the number of states of limited total energy and spatial extension should be finite because of the uncertainty principle. Roughly speaking, it should be proportional to the volume of phase space occupied by the states.

The mathematical formulation of this idea proved to be difficult, however, since phase space is an ambiguous concept in quantum field theory. A first and important step was taken by Haag and Swieca who, starting from a tentative particle interpretation of localized states, were led to the formulation of a compactness criterion [9]. We state here this condition in an equivalent but mathematically somewhat more convenient form. For this purpose we consider for any given bounded spacetime region $\mathcal{O}$ and number $\beta > 0$ the map $\Theta_{\beta,\mathcal{O}}^{(0)}$ from the C^*-algebra $\mathfrak{A}(\mathcal{O})$ of observables localized in $\mathcal{O}$ [14] into the physical Hilbert space $\mathcal{H}$, which is given by

$$\Theta_{\beta,\mathcal{O}}^{(0)}(A) = e^{-\beta H} A\Omega , \quad A \in \mathfrak{A}(\mathcal{O}) . \tag{1.1}$$

Here H denotes the (positive) Hamiltonian and $\Omega \in \mathcal{H}$ the vacuum vector. Each map $\Theta_{\beta,\mathcal{O}}^{(0)}$ is linear and bounded (as a map between the Banach spaces $\mathfrak{A}(\mathcal{O})$ and $\mathcal{H}$). The criterion of Haag and Swieca then amounts to saying that in theories with a reasonable particle interpretation these maps should be *compact*. (Cf. the subsequent section for basic concepts and results pertaining to such maps of "almost finite rank".)

Haag and Swieca proved that their criterion is satisfied in massive free field theory, but fails to hold in some generalized free field theories which do not have a particle interpretation. In the meantime their criterion has also been established in massless free field theories [15] and in some interacting theories in two dimensions [3, Sec. 4]. It provides a characterization of theories with not too many local degrees of freedom.

In spite of its physical significance, the compactness criterion did not prove to be the desired tool for a detailed structural analysis of quantum field theories, the main reason being that the criterion is only of a qualitative nature. Haag and Swieca proposed also a quantitative version of their criterion, but their bounds on the approximate dimension of the pertinent maps were too conservative since they did not take into account the constraints coming from particle statistics. It took almost twenty years until the relevance of the latter point was recognized.

Starting from another heuristical input based on thermodynamical considerations, Buchholz and Wichmann [1] came to the conclusion that in theories admitting thermal equilibrium states the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ should not only be compact but also nuclear. Moreover, they argued that the nuclear norm of these maps should, for small β and large $\mathcal{O}$, coincide with the partition function of the canonical ensemble at

temperature β^{-1} in a container of size corresponding to $\mathcal{O}$. They also showed that their nuclearity criterion is satisfied in free field theory, cf. also [15]. It soon became clear that the properties of the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ can also be expressed in terms of other mathematical concepts, such as the notions of ε-content, approximate dimension, p-nuclearity etc. [3]. The corresponding reformulations of the nuclearity criterion are frequently more convenient in applications. Although they are not completely equivalent, they impose essentially the same physical constraints on the underlying theories.

Another, in a sense dual criterion testing the phase space properties of relativistic quantum field theories has been proposed by Fredenhagen and Hertel [16], cf. also [17]. Instead of considering local excitations of the vacuum whose energy is then cut off as in relation (1.1), these authors proceed from states (positive linear and normalized functionals on the algebra of all observables) with limited energy. They argue that, in analogy to the Haag–Swieca compactness criterion, the restrictions of these states to any given local algebra $\mathfrak{A}(\mathcal{O})$ should form a compact subset of the dual space of $\mathfrak{A}(\mathcal{O})$ in all theories of physical interest. It was later shown by Buchholz and Porrmann that also in this criterion the condition of compactness may be strengthened to nuclearity [17]. These authors also clarified the relation between the various existing formulations of compactness and nuclearity conditions. Roughly speaking, the conditions of Fredenhagen–Hertel type impose somewhat stronger constraints on the phase space properties of a theory than their counterparts in terms of the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$. Cf. [17] for precise statements.

The compactness and nuclearity criterions are normally stated in terms of the local observables of a theory. But they can likewise be posed for the local charge–carrying fields. One then imposes compactness or nuclearity conditions on the maps $\Theta_{\beta,\mathcal{O}}$ which map the C^*-algebra $\mathcal{F}(\mathcal{O})$ of fields localized in $\mathcal{O}$ into the physical Hilbert space $\mathcal{H}$ according to relation (1.1), where $\mathfrak{A}(\mathcal{O})$ then has to be replaced by the algebra $\mathcal{F}(\mathcal{O})$. Since the local field algebras contain the observable algebras as subalgebras, the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ can be recovered from $\Theta_{\beta,\mathcal{O}}$ by restriction, $\Theta_{\beta,\mathcal{O}}^{(0)} = \Theta_{\beta,\mathcal{O}} \upharpoonright \mathfrak{A}(\mathcal{O})$. It follows that compactness or nuclearity conditions on the maps $\Theta_{\beta,\mathcal{O}}$ imply that the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ also have the respective properties.

It is the aim of the present article to study the opposite problem, namely the question of whether one can deduce from compactness or nuclearity properties of the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ corresponding properties of the maps $\Theta_{\beta,\mathcal{O}}$. This question is, on one hand, of interest for a deeper understanding of the relation between the canonical and grand canonical ensembles. According to the thermodynamical considerations in [1], the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ should provide information on the properties of the canonical ensemble of states carrying the charge quantum numbers of the vacuum, whereas

$\Theta_{\beta,\mathcal{O}}$ is linked to the grand canonical ensemble of states of arbitrary charge and zero chemical potential. As was already mentioned, the nuclear norms of these maps are a substitute for the partition functions of the respective ensembles.

A clarification of the relation between the compactness or nuclearity properties of these maps would, on the other hand, allow to deduce algebraic properties of charged (unobservable) fields, such as the split property [18], from phase space properties of the observables. It would thereby lead to a deeper understanding of the physical significance of these algebraic structures which seem to be of importance, e.g., in connection with the formulation of a quantum Noether theorem [19, 20].

In the present investigation we start from two general assumptions on the theory which can be expressed purely in terms of observables. The first assumption concerns the superselection structure: let Σ be some index set labelling the superselection sectors describing localizable charges [21], let d_σ be the statistical dimension of the sector $\sigma \in \Sigma$ and let m_σ be the lowest mass value in this sector (cf. Sec. 3 for precise definitions). Then we assume that

$$\sum_{\sigma \in \Sigma} d_\sigma e^{-\lambda m_\sigma} < \infty \quad \text{for all } \lambda > 0 . \tag{1.2}$$

This condition has clearly to be satisfied if the grand canonical ensemble with zero chemical potential is to exist for arbitrary positive temperatures.

Our second assumption concerns the phase space properties of bilocal excitations of the vacuum which are induced by observables. To this end we consider the appropriately defined algebra $\mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2)$ of bilocal operations in the spacelike separated regions $\mathcal{O}_1$, $\mathcal{O}_2$ and the corresponding maps $\Theta_{\beta,\mathcal{O}_1 \cup \mathcal{O}_2}^{(0)}$. We assume that the compactness or nuclearity properties of these maps are not affected if one keeps $\mathcal{O}_1$, say, fixed and shifts $\mathcal{O}_2$ to spacelike infinity (cf. Sec. 3 for the precise formulation of this condition).

The physical significance of this condition can be understood by appealing again to the thermodynamical interpretation of the pertinent maps. According to that interpretation the map $\Theta_{\beta,\mathcal{O}_1 \cup \mathcal{O}_2}^{(0)}$ is related to the canonical ensemble, confined in two containers of size proportional to $\mathcal{O}_1$, respectively $\mathcal{O}_2$, which are connected by a thin tube. The tube has the effect that, although the total charge of the ensemble is fixed, the charges in the individual containers can fluctuate. This is the familiar situation discussed in the passage from the canonical to the grand canonical ensemble. Our second assumption thus expresses the idea that a separation of the containers should have no significant effects, i.e., the bilocalized ensemble should stabilize. Again, this condition seems to be necessary if the grand canonical ensemble is to exist.

Starting from these assumptions we are able to establish qualitative and quantitative compactness properties of the maps $\Theta_{\beta,\mathcal{O}}$ from corresponding properties of the maps $\Theta^{(0)}_{\beta,\mathcal{O}_1\cup\mathcal{O}_2}$. In particular we will show that if the maps $\Theta^{(0)}_{\beta,\mathcal{O}_1\cup\mathcal{O}_2}$ are of type s (cf. Sec. 2), then the maps $\Theta_{\beta,\mathcal{O}}$ have this property, too, and consequently the field algebra has the split property [3]. Thus the present results reveal an intimate relation between the phase space properties of charged fields and the underlying observables. It is an interesting question whether the additional assumptions on the observables, mentioned above, are necessary to establish these results. Yet this problem is not touched upon in the present investigation.

Our paper is organized as follows. In Sec. 2 we recall basic concepts from the theory of compact linear maps between Banach spaces and establish some useful technical results. Section 3 contains a discussion of the general setting of algebraic quantum field theory as well as of the more specific assumptions entering into our analysis. There we also obtain a preliminary result on some relevant map, following from the cluster theorem, which will allow us to establish our main results in Sec. 4. Some physically significant applications are outlined in the concluding Section 5.

2. Compact maps

In the present investigation we have to rely on various concepts and results from the theory of compact linear maps between Banach spaces. Standard references on this subject are the books [22, 23]. We recall here the basic definitions and collect some useful results. The expert reader may skip this section and return to it later for some technical details.

We begin by explaining our notation. Let $\mathcal{E}$ be any (real or complex) Banach space with norm $\|\cdot\|_{\mathcal{E}}$. The unit ball of $\mathcal{E}$ is denoted by $\mathcal{E}_1$ and the space of continuous linear functionals on $\mathcal{E}$ by $\mathcal{E}^*$. Given another Banach space $\mathcal{F}$, we denote the space of continuous linear maps L from $\mathcal{E}$ to $\mathcal{F}$ by $\mathcal{L}(\mathcal{E},\mathcal{F})$. The latter space is again a Banach space with norm given by

$$\|L\| \doteq \sup\{\|L(E)\|_{\mathcal{F}} : E \in \mathcal{E}_1\} \ . \tag{2.1}$$

A map $L \in \mathcal{L}(\mathcal{E},\mathcal{F})$ is said to be compact if the image of $\mathcal{E}_1$ under the action of L is a precompact subset of $\mathcal{F}$. A convenient measure which provides more detailed information on the size of the range of compact maps is the notion of ε-content.

Definition: Let $L \in \mathcal{L}(\mathcal{E},\mathcal{F})$ be a compact map and let, for given $\varepsilon > 0$, $N_L(\varepsilon)$ be the maximal number of elements $E_i \in \mathcal{E}_1$, $i = 1, \cdots N_L(\varepsilon)$ such that $\|L(E_i - E_k)\| > \varepsilon$ if $i \neq k$. The number $N_L(\varepsilon)$ is called the ε-*content* of L. (Note that $N_L(\varepsilon)$ is finite for all $\varepsilon > 0$ iff L is compact [22].)

It is clear that the ε-content of L increases if ε decreases, and it tends to infinity if ε approaches 0 (unless L is the zero map). If the rank of L is equal to $n \in \mathbb{N}$ (i.e., if the range of L is an n-dimensional subspace of $\mathcal{F}$), then the ε-content $N_L(\varepsilon)$ behaves for small ε like ε^{-n}. The maps L which will appear in our investigation have ε-contents which behave, for small ε, typically like $e^{(M/\varepsilon)^q}$, where M and q are positive numbers. This fact suggests to introduce the quantities (provided they exist)

$$q_L = \limsup_{\varepsilon \searrow 0} \frac{ln\, ln\, N_L(\varepsilon)}{ln\, 1/\varepsilon} \,, \tag{2.2}$$

called the *order* of L, and

$$M_L(q) = \sup_{\varepsilon > 0} \varepsilon \big(ln\, N_L(\varepsilon)\big)^{1/q} \,, \quad q > 0 \,. \tag{2.3}$$

It is noteworthy that q_L is just the infimum of all q for which $M_L(q)$ is finite.

Another way of describing the detailed properties of a compact map is based on the idea to check how well the image of the unit ball $\mathcal{E}_1$ under the action of the map fits into suitable finite dimensional subspaces $\mathcal{S} \subset \mathcal{F}$.

Definition: Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ be a compact map and let $n \in \mathbb{N}_0$. The number

$$\delta_L(n) \doteq \inf\big\{\delta > 0 : L(\mathcal{E}_1) \subset \mathcal{S} + \delta \cdot \mathcal{F}_1 \,, \mathcal{S} \subset \mathcal{F} \,, \dim \mathcal{S} \leq n\big\}$$

is called the *n-th diameter* (of the image of $\mathcal{E}_1$ under the action) of L. In particular $\delta_L(0) = \|L\|$.

A closely related idea is to measure how much a given map deviates from a map of finite rank. The relevant concepts are given in the following definition.

Definition: Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$.
i) For given $n \in \mathbb{N}_0$, the number

$$\alpha_L(n) \doteq \inf\big\{\|L - L_n\| : L_n \in \mathcal{L}(\mathcal{E}, \mathcal{F}) \,, \text{ rank } L_n \leq n\big\}$$

is called the *n-th approximation number* of L. In particular $\alpha_L(0) = \|L\|$.
ii) The map L is said to be of *type l^p*, $p > 0$, if

$$[]L[]_p = \Big(\sum_n \alpha_L(n)^p\Big)^{1/p} < \infty \,.$$

(As is indicated by the notation, the maps of type l^p form, for fixed p, a linear space, and $[] \cdot []_p$ is a (quasi) norm on this space, cf. [22].)

It is apparent that $\delta_L(n) \leq \alpha_L(n)$, but in general there does not hold equality. As a matter of fact, it need not even be true that $\alpha_L(n)$ tends to 0 for large n if $\delta_L(n)$ does, i.e., if L is a compact map. The two quantities coincide, however, in the important special case where $\mathcal{F}$ is a Hilbert space.

Lemma 2.1. *Let $\mathcal{F}$ be a Hilbert space and let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$. Then $\alpha_L(n) = \delta_L(n)$, $n \in \mathbb{N}_0$.*

Proof: As already mentioned, there holds always $\delta_L(n) \leq \alpha_L(n)$. The reverse inequality follows from the fact that for any n-dimensional subspace $\mathcal{S}$ of a Hilbert space $\mathcal{F}$ there exists an orthogonal projection $P_\mathcal{S} \in \mathcal{L}(\mathcal{F}, \mathcal{F})$ of rank n which projects onto $\mathcal{S}$. Hence

$$\alpha_L(n) \leq \inf\big\{\|L - P_\mathcal{S} \cdot L\| : \mathcal{S} \subset \mathcal{F}, \dim \mathcal{S} = n\big\} = \delta_L(n) \ ,$$

where the last equality follows from the fact that $\|L(E) - P_\mathcal{S} \cdot L(E)\|_\mathcal{F} = \inf\big\{\|L(E) - F\|_\mathcal{F} : F \in \mathcal{S}\big\}$. $\qquad\square$

We conclude our list of quantities which measure the properties of compact maps with still another concept. In this variant one studies the decompositions of a given map into the most elementary ones, the maps of rank one, which are of the form $e(\cdot)F$, where $e \in \mathcal{E}^*$ and $F \in \mathcal{F}$.

Definition: Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ be a map such that for suitable sequences $e_n \in \mathcal{E}^*$, $F_n \in \mathcal{F}$, $n \in \mathbb{N}$ there holds (in the sense of strong convergence)

$$L(E) = \sum_n e_n(E) F_n \ , \ E \in \mathcal{E} \ .$$

If in addition there holds $\sum_n \|e_n\|_{\mathcal{E}^*}^p \|F_n\|_\mathcal{F}^p < \infty$ for some $p > 0$ the map L is said to be *p-nuclear*. The space of p-nuclear maps is equipped with the (quasi) norm [22]

$$\|L\|_p = \inf\Big(\sum_n \|e_n\|_{\mathcal{E}^*}^p \|F_n\|_\mathcal{F}^p\Big)^{1/p} \ ,$$

where the infimum is to be taken with respect to all possible decompositions of L. We note that 1-nuclear maps are in general called nuclear maps.

The various notions mentioned above are related to each other, although these relations are not very rigid. In the present investigation the concept of ε-content will prove to be the most useful one. Since the other notions are also frequently used in the literature, we collect here some useful information about their respective relations. Again we restrict attention to the case where $\mathcal{F}$ is a Hilbert space (cf. [3; Sec. 2] for the general case). The following lemma is a slight improvement on the general results in [22; Sec. 9.6] for the special case at hand.

Lemma 2.2. *Let $\mathcal{F}$ be a Hilbert space, let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ be a compact map, and let $\varepsilon > 0$. There hold for $m \in \mathbb{N}$ and sufficiently large $n \in \mathbb{N}$ such that $\alpha_L(n) < \varepsilon/2$ the inequalities*

$$\frac{2^m \alpha_L(0) \cdots \alpha_L(m-1)}{v_m m! \, \varepsilon^m} \leq N_L(\varepsilon) \leq \left(\frac{2\alpha_L(0)}{\varepsilon - 2\alpha_L(n)} + 1 \right)^n$$

where v_m is the volume of the unit ball in $\mathbb{R}^m$.

Proof: We discuss here only the case where $\mathcal{F}$ is a real Hilbert space, the argument for complex Hilbert spaces is analogous. To verify the statement we have to study the geometric properties of the set $\mathcal{S} \doteq L(\mathcal{E}_1) \subset \mathcal{F}$. Since $\mathcal{E}_1$ is absolutely convex and L is linear and compact, the set $\mathcal{S}$ is also absolutely convex and precompact. In order to prove the first inequality we fix some $0 < \delta < 1$ and pick vectors $\Phi_m \in \mathcal{S}$, $m \in \mathbb{N}_0$, such that $\|(1 - P_m)\Phi_m\|_{\mathcal{F}} = (1-\delta)\alpha_L(m)$, where P_m is the orthogonal projection onto the subspace generated by $\Phi_0, \ldots \Phi_{m-1}$, and $P_0 = 0$. Because of absolute convexity, $\mathcal{S}$ contains the polygon $\mathcal{S}_m = \left\{ \sum_{i=0}^{m-1} c_i \Phi i : \sum_{i=0}^{m-1} |c_i| \leq 1 \right\}$. Let N_m be the maximal number of vectors $\Psi_j \in \mathcal{S}_m$, $j = 1, \cdots N_m$ with mutual distances larger than ε. In order to get a lower bound on N_m we regard $\mathcal{S}_m$ as a subset of $\mathbb{R}^m$ and note that the collection of m-balls of radius ε which are centered at the points Ψ_j, $j = 1, \cdots N_m$ provides a covering of $\mathcal{S}_m$. The volume of the polygon $\mathcal{S}_m$ is equal to $2^m (1 - \delta)^m \alpha_L(0) \cdots \alpha_L(m-1)/m!$. It is not larger than $N_m \varepsilon^m v_m$, i.e., the volume of the N_m balls covering it. But $N_m \leq N_L(\varepsilon)$, hence the first inequality in the statement follows since δ can be made arbitrarily small.

To prove the second inequality we recall that $\alpha_L(n) = \delta_L(n)$ since $\mathcal{F}$ is a Hilbert space. Thus, given $\delta > 0$, there is some at most n-dimensional subspace $\mathcal{S}_n \subseteq \mathcal{F}$ and a corresponding orthogonal projection P_n such that $\|(1 - P_n)\Phi\|_{\mathcal{F}} \leq (1+\delta)\alpha_L(n)$ for all $\Phi \in \mathcal{S}$. Hence if there are N vectors $\Phi_i \in \mathcal{S}$, $i = 1, \cdots N$, such that $\|\Phi_i - \Phi_k\|_{\mathcal{F}} > \varepsilon$ for $i \neq k$, there holds $\|E_n \Phi_i - E_n \Phi_k\|_{\mathcal{F}} > \varepsilon - 2(1+\delta)\alpha_L(n) = \varepsilon'$. If n is such that $\varepsilon' > 0$ we conclude, since $\|E_n \Phi\|_{\mathcal{F}} \leq \|\Phi\|_{\mathcal{F}} \leq \alpha_L(0)$ for $\Phi \in \mathcal{S}$, that there are not less than N elements in the set $E_n \cdot \mathcal{S} \subset \alpha_L(0)(\mathcal{S}_n)_1$ with mutual distance larger than ε'. Hence there are not less than N disjoint n-balls of radius $\varepsilon'/2$ fitting into the n-ball $\left(\alpha_L(0) + \varepsilon'/2 \right) \cdot (\mathcal{S}_n)_1$. Comparing volumes as in the preceding step and making δ arbitrarily small one arrives at the second inequality. $\qquad\square$

In the analysis of the compactness properties of maps it is often most convenient to determine their nuclear (quasi) norms. The problem is then to infer from the size of these norms on the ε-content and vice versa. In order to establish such a relation we need the following two lemmas which are of interest in their own right.

Lemma 2.3. *Let $L_1, \ldots L_n \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ be compact maps with ε-content $N_{L_1}(\varepsilon)$, $\cdots N_{L_n}(\varepsilon)$, respectively. Then the ε-content of the map $L_1 + \cdots + L_n$ is bounded by*

$$N_{L_1+\cdots+L_n}(\varepsilon) \leq \inf_{\varepsilon_1+\cdots+\varepsilon_n=\varepsilon/2} N_{L_1}(\varepsilon_1) \cdots N_{L_n}(\varepsilon_n) \ .$$

Proof: According to the very definition of ε-content there exist for given $\varepsilon_m > 0$, $m = 1, \cdots n$, exactly $N_{L_m}(\varepsilon_m)$ elements $E_{m,k} \in \mathcal{E}_1$, $k = 1, \cdots N_{L_m}(\varepsilon_m)$, such that for any $E \in \mathcal{E}_1$ there holds

$$\|L_m(E - E_{m,k_m(E)})\|_{\mathcal{F}} \leq \varepsilon_m$$

for some suitable index $k_m(E)$. Let $\mathbb{I}$ be the set of all n-tuples $(k_1(E), \cdots k_n(E))$, $E \in \mathcal{E}_1$, which appear in this way. The cardinality of $\mathbb{I}$ is less than or equal to $N_{L_1}(\varepsilon_1) \cdots N_{L_n}(\varepsilon_n)$. For each $(i_1, \cdots i_n) \in \mathbb{I}$ we pick some operator $\bar{E}_{i_1, \cdots i_n} \in \mathcal{E}_1$ such that $\|L_m(\bar{E}_{i_1, \cdots i_n} - E_{m,i_m})\|_{\mathcal{F}} \leq \varepsilon_m$ for $m = 1, \ldots n$. Then there holds for any $E \in \mathcal{E}_1$

$$\|(L_1 + \cdots + L_n)(E - \bar{E}_{i_1, \cdots i_n})\|_{\mathcal{F}} \leq \sum_{m=1}^{n} \|L_m(\bar{E} - E_{i_1, \cdots i_n})\|_{\mathcal{F}}$$

$$\leq \sum_{m=1}^{n} \|L_m(E - E_{m,i_m})\|_{\mathcal{F}} + \sum_{m=1}^{n} \|L_m(E_{m,i_m} - \bar{E}_{i_1, \cdots i_n})\|_{\mathcal{F}} \ .$$

Consequently there exists some index $(i_1, \cdots i_n) \in \mathbb{I}$ such that

$$\|(L_1 + \cdots + L_n)(E - \bar{E}_{i_1, \cdots i_n})\|_{\mathcal{F}} \leq 2(\varepsilon_1 + \cdots + \varepsilon_n) \ .$$

Setting $\varepsilon_1 + \cdots + \varepsilon_n = \varepsilon/2$ we conclude that the ε-content of the map $(L_1 + \cdots + L_n)$ cannot be larger than the cardinality of $\mathbb{I}$, so the statement follows. $\square$

Lemma 2.4. *Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ and let $L_n \in \mathcal{L}(\mathcal{E}, \mathcal{F}_n)$, $n \in \mathbb{N}$, be a sequence of maps which dominates L asymptotically, i.e., the sequences $\|L_n(E)\|_{\mathcal{F}_n}, E \in \mathcal{E}$, converge and*

$$\lim_n \|L_n(E)\|_{\mathcal{F}_n} \geq \|L(E)\|_{\mathcal{F}} \ , \ E \in \mathcal{E} \ .$$

If all maps L_n are compact and have uniformly bounded ε-contents, then L is also compact, and

$$N_L(\varepsilon) \leq \liminf_n N_{L_n}(\varepsilon) \ , \ \varepsilon > 0 \ .$$

Proof: Let $E_i \in \mathcal{E}_i$, $i = 1, \cdots N$ be such that $\|L(E_i) - L(E_k)\|_{\mathcal{F}} > \varepsilon$ for $i \neq k$. Since the sequence $L_n, n \in \mathbb{N}$ dominates L asymptotically there exists some n_0 such that for $n \geq n_0$ there holds $\|L_n(E_i) - L_n(E_k)\|_{\mathcal{F}_n} > \varepsilon$ if $i \neq k$. Hence $N \leq N_{L_n}(\varepsilon)$ for all $n \geq n_0$ and consequently $N \leq \liminf_n N_{L_n}(\varepsilon)$. $\square$

In the following proposition, which is partly taken from [22], some useful relations between the ε-content (respectively the related quantity $M_L(q)$ introduced in (2.3)), the l^p-norms and the nuclear norms of a map L are established.

Proposition 2.5. *Let $\mathcal{F}$ be a Hilbert space and let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$. There hold the inequalities*

  i) $\|L\|_p \leq a_p \cdot []L[]_p \quad for \quad 0 < p \leq 1$

  ii) $[]L[]_p \leq b_{p,q} \cdot M_L(q) \quad for \quad p > 2q/(2-q) > 0$

  iii) $M_L(q) \leq c_{p,q} \cdot \|L\|_p \quad for \quad q > p/(1-p) > 0.$

Here a_p, $b_{p,q}$ and $c_{p,q}$ are numerical constants which do not depend on L.

Proof: i) The first inequality has been established in [22, Prop. 8.4.2]. As a matter of fact it holds for maps between arbitrary Banach spaces. For the constant in the inequality one has the upper bound $a_p \leq 2^{2+3/p}$.

ii) According to the definition of $M_L(q)$ there holds $N_L(\varepsilon) \leq e^{(M_L(q)/\varepsilon)^q}$ and consequently

$$\inf_{\varepsilon > 0} \varepsilon^n N_L(\varepsilon) \leq \left(\frac{q^{1/q} e^{1/q} M_L(q)}{n^{1/q}} \right)^n \quad , \; n \in \mathbb{N} \; .$$

Making use of the first inequality in Lemma 2.2 and the fact that the approximation numbers are monotonously decreasing, we conclude that

$$\alpha_L(n-1) \leq \frac{(v_n n!)^{1/n}}{2} \frac{q^{1/q} e^{1/q}}{n^{1/q}} \cdot M_L(q), \; n \in \mathbb{N} \; .$$

But $(v_n n!)^{1/n} \leq \sqrt{2\pi n}$, so the statement follows after summation. Moreover, one obtains a bound on the constant $b_{p,q}$ given by $b_{p,q} \leq \sqrt{\frac{\pi}{2}} q^{1/q} e^{1/q} \cdot \left(\frac{p}{p - 2q/(2-q)} \right)^{1/p}$.

iii) For given $\delta > 0$ there exists a sequence of maps $L_n \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, $n \in \mathbb{N}$, of rank one such that $L(E) = \sum_n L_n(E)$, $E \in \mathcal{E}_1$, in the sense of strong convergence, and $\left(\sum_n \|L_n\|^p \right)^{1/p} \leq (1+\delta) \cdot \|L\|_p$. We assume in the following that the sequence of norms $\|L_n\|$ is monotonically decreasing. (This can always be accomplished by reordering of the operators L_n since the sum in the decomposition of L is absolutely converging for p-nuclear operators, $0 < p \leq 1$.) It then follows that for any given $n \in \mathbb{N}$ there holds $n \cdot \|L_n\|^p \leq \sum_{m=1}^n \|L_m\|^p \leq (1+\delta)^p \|L\|_p^p$ and consequently

$$\|L_n\| \leq \frac{(1+\delta)\|L\|_p}{n^{1/p}} \quad , \quad n \in \mathbb{N} \; .$$

Since the maps L_n are of rank 1 it is straightforward to estimate their ε-contents $N_{L_n}(\varepsilon)$. In the real linear case $N_{L_n}(\varepsilon)$ is not larger than $\max(4\|L_n\|/\varepsilon, 1)$, and in the complex linear case than $\max(4^2\|L_n\|^2/\varepsilon^2, 1)$. We treat in the following the latter

case, the former one is analogous. Applying Lemma 2.3 to the map $(L_1 + \cdots + L_n)$ we obtain the bound

$$N_{L_1 + \cdots + L_n}(\varepsilon) \leq N_{L_1}(\varepsilon_1) \cdots N_{L_n}(\varepsilon_n) \ ,$$

provided $\varepsilon_1 + \cdots + \varepsilon_n = \frac{\varepsilon}{2}$. This constraint on $\varepsilon_1, \cdots \varepsilon_n$ can be relaxed to $\varepsilon_1 + \cdots + \varepsilon_n \leq \frac{\varepsilon}{2}$ since the ε-contents increase if ε decreases.

We fix r, $1 < r < 1/p$, and set

$$\varepsilon_i = \left(\frac{r}{r-1}(1 + \delta)^{pr} \|L\|_p^{pr} \right)^{-1} \|L_i\|^{pr} \cdot \frac{\varepsilon}{2} \ , \quad i = 1, \cdots n \ .$$

In view of the preceding bound on the norms $\|L_i\|$ there holds $(\varepsilon_1 + \cdots + \varepsilon_n) \leq \varepsilon/2$. Moreover, the numbers $4\|L_m\|/\varepsilon_m$, $m = 1, \cdots n$, are monotonically decreasing. Let n_* be the largest index for which $4\|L_{n_*}\|/\varepsilon_{n_*} > 1$; if there is no such index we put $n_* = n$. Since for $m > n_*$ there holds $N_{L_m}(\varepsilon_m) = 1$ one gets

$$N_{L_1 + \cdots + L_n}(\varepsilon) \leq$$

$$\leq \varepsilon^{-2n_*} \cdot 4^{3n_*} \left(\frac{r}{r-1}(1+\delta)^{pr}\|L\|_p^{pr} \right)^{2n_*} \cdot \prod_{m=1}^{n_*} \|L_m\|^{2(1-pr)} \leq$$

$$\leq \varepsilon^{-2n_*} \cdot (n_*)!^{2\frac{pr-1}{p}} \left(8\frac{r}{r-1}(1+\delta)\|L\|_p \right)^{2n_*} .$$

Taking the supremum of this upper bound with respect to $n_* \in \mathbb{N}$ one finds after a straightforward calculation that

$$N_{L_1 + \cdots + L_n}(\varepsilon) \leq e^{(c_{p,q} \cdot (1+\delta)\|L\|_p / \varepsilon)^q}$$

where $q = p/(1 - pr)$ and $c_{p,q} \leq 8(2e/q)^{1/q} \frac{1}{1-pq/(q-p)}$. Since the sequence of maps $(L_1 + \cdots + L_n) \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, $n \in \mathbb{N}$, converges strongly to L, it dominates L asymptotically in the sense of Lemma 2.4. Hence the preceding bound holds also for $N_L(\varepsilon)$. The statement then follows in the limit of arbitrarily small δ. (Note that for $1 < r < 1/p$ the corresponding parameter q runs through the given range.) $\qquad \square$

It follows from this result that the maps of order 0 coincide with the maps which are of type l^p or p-nuclear for all $p > 0$. Such maps are said to be of *type s*.

We conclude this section with two results on precompact sets which will be needed later. In these statements there appears the notion of the ε-content $N^{\mathcal{S}}(\varepsilon)$ of a subset $\mathcal{S}$ of a Banach space $\mathcal{E}$: it is the largest natural number of elements $S_i \in \mathcal{S}$, $i = 1, \cdots N^{\mathcal{S}}(\varepsilon)$, such that $\|S_i - S_k\|_{\mathcal{E}} > \varepsilon$ for $i \neq k$.

Lemma 2.6. *Let $\mathcal{E}_n$ be an n-dimensional complex Hilbert space, $n \in \mathbb{N}$, and let $\mathcal{B} \subset \mathcal{L}(\mathcal{E}_n , \mathcal{E}_n)$ be a subset of operators of norm not larger than $b > 0$. Then the ε-content of $\mathcal{B}$ is bounded by*

$$N^{\mathcal{B}}(\varepsilon) \leq \left(\frac{2nb}{\varepsilon} + 1 \right)^{n^2} .$$

Proof: Let $\Phi_i \in \mathcal{E}_n$, $i = 1, \cdots n$ be an orthonormal basis. There holds for any operator $B \in \mathcal{L}(\mathcal{E}_n , \mathcal{E}_n)$

$$\|B\| \leq \left(\sum_{i,k} |(\Phi_i , B\Phi_k)|^2 \right)^{1/2} \leq n\|B\| .$$

Hence, viewing the matrix elements $(\Phi_i, B\Phi_k)$, $i, k = 1, \cdots n$ as components of a vector in $\mathbb{C}^{n^2}$ we conclude that the ε-content of $\mathcal{B}$ is not larger than the ε-content of a ball of radius nb in $\mathbb{C}^{n^2}$. The bound on $N^{\mathcal{B}}(\varepsilon)$ then follows by comparing volumes as in the proof of Lemma 2.2. $\qquad\square$

Lemma 2.7. *Let $\mathcal{E}$ be a Banach space, let $\mathcal{I} \subset \mathcal{L}(\mathcal{E}, \mathcal{E})$ be a compact set of isometries which is stable under taking inverses, and let $\mathcal{X}$ be a subset of $\mathcal{E}$ with the following property: for each $\varepsilon > 0$ there exist at most $N(\varepsilon)$ elements $X_i \in \mathcal{X}$, $i = 1, \cdots N(\varepsilon)$, such that*

$$\inf_{I \in \mathcal{I}} \|X_i - I(X_k)\|_{\mathcal{E}} > \varepsilon \ , \ i \neq k \ .$$

Then $\mathcal{X}$ is precompact and its ε-content is bounded by

$$N^{\mathcal{X}}(\varepsilon) \leq \inf_{\varepsilon_1 + x\varepsilon_2 = \frac{\varepsilon}{2}} N(\varepsilon_1) N^{\mathcal{I}}(\varepsilon_2) ,$$

where $x = \sup_{X \in \mathcal{X}} \|X\|_{\mathcal{E}}$ and $N^{\mathcal{I}}(\varepsilon)$ is the ε-content of $\mathcal{I}$.

Proof: By assumption there exist for given $\varepsilon_1 > 0$ $N(\varepsilon_1)$ elements $X_i \in \mathcal{X}$, $i = 1, \cdots N(\varepsilon_1)$, such that for any $Y \in \mathcal{X}$ there holds either $\inf_{I \in \mathcal{I}} \|Y - I(X_i)\|_{\mathcal{E}} \leq \varepsilon_1$ or $\inf_{I \in \mathcal{I}} \|X_i - I(Y)\|_{\mathcal{E}} \leq \varepsilon_1$ for some suitable index i. Since $\mathcal{I}$ consists of isometries and is stable under taking inverses, we may assume without loss of generality that the first inequality holds. Moreover, since $\mathcal{I}$ is compact, there exists also some $I_Y \in \mathcal{I}$ such that $\|Y - I_Y(X_i)\|_{\mathcal{E}} \leq \varepsilon_1$. It follows in particular that $\|Y\|_{\mathcal{E}} \leq \sup_i \|X_i\|_{\mathcal{E}} + \varepsilon_1$, hence the set $\mathcal{X}$ is bounded, $x = \sup_{X \in \mathcal{X}} \|X\|_{\mathcal{E}} < \infty.$

Let $\varepsilon_2 > 0$. Then there exist $N^{\mathcal{I}}(\varepsilon_2)$ elements I_k , $k = 1, \cdots N^{\mathcal{I}}(\varepsilon_2)$ such that for any $I \in \mathcal{I}$ there holds $\|I - I_k\| \leq \varepsilon_2$ for some suitable k. Hence we have for any $Y \in \mathcal{X}$ the estimate

$$\|Y - I_k(X_i)\|_{\mathcal{E}} \leq \|Y - I_Y(X_i)\|_{\mathcal{E}} + x\|I_Y - I_k\| \leq \varepsilon_1 + x\varepsilon_2$$

for a suitable index pair i, k. We proceed now as in the proof of Lemma 2.4: let $\mathbb{I}$ be the set of indexes (i, k) which appear in this way if Y runs through $\mathcal{X}$. For each such index we pick $Y_{ik} \in \mathcal{X}$ such that $\|Y_{ik} - I_k(X_i)\|_{\mathcal{E}} \leq \varepsilon_1 + x\varepsilon_2$. It follows that for any $Y \in \mathcal{X}$ there exists some $(i, k) \in \mathbb{I}$ such that $\|Y - Y_{ik}\|_{\mathcal{E}} \leq 2(\varepsilon_1 + x\varepsilon_2)$. Setting $2(\varepsilon_1 + x\varepsilon_2) = \varepsilon$ and noting that the cardinality of $\mathbb{I}$ is at most $N(\varepsilon_1)N^{\mathcal{I}}(\varepsilon_2)$ the statement follows. □

3. Observables, fields and compactness conditions

We list in this section the standard assumptions made in the theory of local observables [14] and formulate in precise terms the more specific conditions indicated in the Introduction. We then recall some fundamental results on the superselection structure and the structure of charge–carrying fields which have been derived from these assumptions by Doplicher and Roberts [24]. Making use of these facts and the cluster theorem we will be able to establish some preliminary results on the compactness properties of certain specific maps which are the basis for the analysis in the subsequent section.

We suppose that the net $\mathfrak{A}$ of local observables is concretely given as an inclusion preserving mapping (hence the term net)

$$\mathcal{O} \longrightarrow \mathfrak{A}(\mathcal{O}), \quad \mathcal{O} \in \mathcal{K} \tag{3.1}$$

from the set of all double cones $\mathcal{K}$ in Minkowski space $\mathbb{R}^4$ to von Neumann algebras $\mathfrak{A}(\mathcal{O})$ acting on the (complex, separable) vacuum Hilbert space $\mathcal{H}_0$.

The net $\mathfrak{A}$ is supposed to be local, i.e.,

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2' , \tag{3.2}$$

where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathcal{B}'$ the set of all bounded operators which commute with a given set $\mathcal{B}$ of bounded operators on the underlying Hilbert space.

We will also have occasion to consider algebras of observables for space–time regions other than double cones. Given $\mathcal{O} \in \mathcal{K}$ we define $\mathfrak{A}(\mathcal{O}')$ as the C^*-algebra which is generated by all algebras $\mathfrak{A}(\mathcal{O}_1)$, $\mathcal{O}_1 \subset \mathcal{O}'$ and $\mathcal{O}_1 \in \mathcal{K}$. Because of locality

there holds $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')'$. But we make here the stronger assumption (Haag duality)

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')' \quad , \quad \mathcal{O} \in \mathcal{K} . \tag{3.3}$$

In the presence of spontaneously broken symmetries the net of observables does not comply with this condition, cf. [25, 26]. One should then identify $\mathfrak{A}$ in the following with the so-called dual net $\mathfrak{A}^d$, $\mathcal{O} \longrightarrow \mathfrak{A}^d(\mathcal{O}) \doteq \mathfrak{A}(\mathcal{O}')'$, which satisfies Haag duality under quite general conditions [27].

We will also consider algebras corresponding to pairs of double cones $\mathcal{O}_1, \mathcal{O}_2$. In the spirit of Haag duality we define $\mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2)$ as the largest algebra of operators which commute with all observables in $\mathcal{O}_1' \cap \mathcal{O}_2'$,

$$\mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2) = \big(\bigcup_{\mathcal{O} \subset \mathcal{O}_1' \cap \mathcal{O}_2'} \mathfrak{A}(\mathcal{O}) \big)' . \tag{3.4}$$

So, strictly speaking, the algebra $\mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2)$ should be regarded as an element of the dual net. But in order to simplify the notation we omit the superscript d since there is no danger of confusion.

We assume that on the vacuum Hilbert space $\mathcal{H}_0$ there exists some continuous unitary representation U_0 of the space–time translations $x = (\underline{x}, t)$ which acts covariantly on the net $\mathfrak{A}$,

$$U_0(x) \, \mathfrak{A}(\mathcal{O}) U_0(x)^{-1} = \mathfrak{A}(\mathcal{O} + x) , \tag{3.5}$$

has spectrum in the closed forward lightcone, $sp\,U_0 \subset \bar{V}_+$, and leaves the (up to a phase unique) vacuum vector $\Omega \in \mathcal{H}_0$ invariant. We also assume that Ω has the Reeh–Schlieder property, i.e., the set of vectors $\mathfrak{A}(\mathcal{O})\Omega$ is dense in $\mathcal{H}_0$ for each $\mathcal{O} \in \mathcal{K}$.

Our first more specific assumption concerns the phase space properties of the theory. It is expressed as follows.

Condition C: Let $\beta > 0$, let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, and let $\Theta^{(0)}_{\beta,\mathcal{O}_1 \cup \mathcal{O}_2}$ be the map from $\mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2)$ into $\mathcal{H}_0$ given by

$$\Theta^{(0)}_{\beta,\mathcal{O}_1 \cup \mathcal{O}_2}(A) \doteq e^{-\beta H_0} A \Omega , \ A \in \mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2) ,$$

where H_0 is the (positive) generator of the time translations on $\mathcal{H}_0$. These maps are compact for any choice of $\mathcal{O}_1, \mathcal{O}_2$. Moreover, if $N_{\Theta_{\beta,\mathcal{O}_1 \cup \mathcal{O}_2}}(\varepsilon)$ denotes the ε-content of these maps there holds for any $\varepsilon > 0$

$$\liminf_{\underline{x}} N_{\Theta^{(0)}_{\beta,\mathcal{O}_1 \cup (\mathcal{O}_2 + \underline{x})}}(\varepsilon) = \underline{N}_{\beta,\mathcal{O}_1,\mathcal{O}_2}(\varepsilon) < \infty ,$$

where the limit is understood for translations $\underline{x}$ tending to spacelike infinity.

This condition is a variant of the compactness criterion of Haag and Swieca. We will later make more restrictive assumptions on the upper bounds $\underline{N}_{\beta,\mathcal{O}_1,\mathcal{O}_2}(\varepsilon)$ appearing in the statement. If one assumes, for example, that the maps $\Theta^{(0)}_{\beta,\mathcal{O}_1\cup(\mathcal{O}_2+\underline{x})}$ are p-nuclear for fixed $0 < p < 1$ and that $\|\Theta^{(0)}_{\beta,\mathcal{O}_1\cup(\mathcal{O}_2+\underline{x})}\|_p \leq const$ for $\underline{x}$ tending to spacelike infinity, it follows from the third statement in Proposition 2.5 that $\underline{N}_{\beta,\mathcal{O}_1,\mathcal{O}_2}(\varepsilon) \leq e^{(M_q/\varepsilon)^q}$ for $q > \frac{p}{1-p}$ and some $M_q < \infty$. Such nuclearity properties can easily be established in free field theory, cf. the appendix in [3]. They are expected to hold quite generally in theories of physical interest.

Our next assumption concerns the charged states of the theory. We are interested here in states of finite energy carrying a localizable charge. As has been discussed in [21], these states can be characterized by a simple "selection criterion", testing their localization properties. They are put together into superselection sectors, i.e., equivalence classes of (pure) states inducing equivalent representations of the observables. We label these sectors by the elements of some index set Σ. There are two physically significant data of the sectors which enter into our second condition. First, each sector $\sigma \in \Sigma$ has an intrinsically defined statistical dimension $d_\sigma \in \mathbb{N}$, which specifies the order of (para-)statistics of the states in the sector; for sectors with Bose or Fermi statistics one has $d_\sigma = 1$ [21]. The second relevant quantity is the threshold mass m_σ, i.e., the lower boundary of the spectrum of the mass operator in the sector σ (which may be positive or 0).

Condition P: For each $\lambda > 0$ there holds

$$\mathcal{Z}(\lambda) \doteq \sum_\sigma d_\sigma e^{-\lambda m_\sigma} < \infty$$

where the sum extends over all sectors $\sigma \in \Sigma$. (The function $\mathcal{Z}$ will be called *little partition function* as it provides a lower bound for the grand partition function for zero chemical potential. Actually it would be sufficient for our analysis if $\mathcal{Z}(\lambda) < \infty$ for some sufficiently small $\lambda > 0$.)

We emphasize that these postulates involve only the observables of a theory, they do not depend on the existence of charge carrying fields. But according to a deep result of Doplicher and Roberts [24] such fields can always be *constructed* under the above conditions. (The special assumptions C and P are not needed there.) More precisely, there is a field net $\mathcal{F}$,

$$\mathcal{O} \longrightarrow \mathcal{F}(\mathcal{O}) \ , \ \mathcal{O} \in \mathcal{K} \tag{3.6}$$

of von Neumann algebras which act on a complex Hilbert space $\mathcal{H}$ containing $\mathcal{H}_0$ as a subspace. The algebras $\mathcal{F}(\mathcal{O})$ are generated by an irreducible set of field operators with normal Bose– and Fermi commutation relations at spacelike distances. On $\mathcal{H}$ there is a continuous unitary representation U of the translations, extending U_0, which satisfies the spectrum condition, $sp\, U \subset \bar{V}_+$, and acts covariantly on the field net,

$$U(x)\mathcal{F}(\mathcal{O})U(x)^{-1} = \mathcal{F}(\mathcal{O} + x) \ . \tag{3.7}$$

Moreover, there is a compact group G and a continuous, unitary and faithful representation V of G on $\mathcal{H}$ which commutes with the translations U and acts locally on the field net,

$$V(g)\mathcal{F}(\mathcal{O})V(g)^{-1} = \mathcal{F}(\mathcal{O}) \ , \ g \in G \ . \tag{3.8}$$

We call G the (global) gauge group and the unitaries $V(g)$, $g \in G$, gauge transformations.

The Hilbert space $\mathcal{H}$ can be decomposed into subspaces $\mathcal{H}^\sigma$, $\sigma \in \Sigma$, on which V acts like a multiple of an irreducible representation of G of dimension d_σ (i.e., the dimension of the respective representation of G coincides with the statistical dimension of the sector σ). The net of fixed points under the gauge group, $\mathcal{O} \longrightarrow \mathcal{F}^G(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \cap V(G)'$, leaves each $\mathcal{H}^\sigma$ invariant and its restriction to $\mathcal{H}^\sigma$ coincides (up to multiplicity) with the irreducible representation of the net of observables $\mathcal{O} \longrightarrow \mathfrak{A}(\mathcal{O})$ in the sector σ. In particular, the vacuum sector $\mathcal{H}_0$ coincides with the set of G-invariant vectors in $\mathcal{H}$ and the restriction of the fixed point net $\mathcal{F}^G$ to $\mathcal{H}_0$ coincides with the original net $\mathfrak{A}$ of observables on $\mathcal{H}_0$. Thus the superselection sectors $\sigma \in \Sigma$ are in one–to–one correspondence to the irreducible representation of G and Σ may therefore be identified with the spectrum (dual) $\hat{G}$ of G.

As discussed in the Introduction, we want to analyse the phase space properties of the field net $\mathcal{F}$ which, as explained, is canonically associated with any local net $\mathfrak{A}$ of observables. To this end we have to study the compactness properties of the maps $\Theta_{\beta,\mathcal{O}}$ from $\mathcal{F}(\mathcal{O})$ to $\mathcal{H}$, given by

$$\Theta_{\beta,\mathcal{O}}(\psi) = e^{-\beta H}\psi\Omega \ , \ \psi \in \mathcal{F}(\mathcal{O}), \tag{3.9}$$

where H is the (positive) generator of time translations on $\mathcal{H}$. In order to carry out this analysis we have to make still another assumption which amounts to the existence of a PCT operator.

Condition J: There is an antiunitary involution J on $\mathcal{H}$ which commutes with the gauge transformations $V(g)$, $g \in G$, satisfies $JU(x) = U(-x)J$ as well as $J\Omega = \Omega$

and acts geometrically on the net,

$$JF(\mathcal{O})J = \mathcal{F}(-\mathcal{O}) \ , \ \mathcal{O} \in \mathcal{K} \ .$$

In contrast to standard quantum field theory, the existence of such a J may not follow from the very general assumptions outlined above [28]. But it seems likely that it can be deduced from a similar condition, involving only the observables, by a combination of results in [29] on the action of the PCT–operator on local morphisms of $\mathfrak{A}$ with the methods developed in [30].

Let us turn now to the investigation of our problem. We begin by introducing some further notations. Let dg be the normalized Haar measure on G and let, for any bounded operator $B \in \mathcal{B}(\mathcal{H})$,

$$M(B) = \int dg V(g) \, B \, V(g)^{-1} \ , \tag{3.10}$$

the mean of B with respect to the gauge group G. The map $M \in \mathcal{L}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ is obviously linear and has norm 1.

The elements of $\mathcal{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, will be denoted by ψ, ψ', and the space–time translated operators ψ by $\psi(x) \doteq U(x)\psi U(x)^{-1}$. We write $\psi(\underline{x})$ if the underlying translation is spacelike and $\psi(t)$ if it is a time translation.

We also consider the direct products $\mathcal{F}(\mathcal{O}_1) \times \mathcal{F}(\mathcal{O}_2)$, $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, and regard them as normed vector spaces with norm given by

$$\|\sum_{i,k} \psi_i^{(1)} \times \psi_k^{(2)}\| = \inf\left\{\sum_{l,m} \|\psi_l^{(1)'}\| \|\psi_m^{(2)'}\| : \sum_{l,m} \psi_l^{(1)'} \times \psi_m^{(2)'} = \sum_{i,k} \psi_i^{(1)} \times \psi_k^{(2)}\right\}$$
$$\tag{3.11}$$

in an obvious notation. By completion we obtain Banach spaces, but this is of no relevance in the sequel.

We fix in the following $\beta > 0$ and some double cone $\mathcal{O} \in \mathcal{K}$ centered at 0, so that $\mathcal{O} = -\mathcal{O}$, and consider a family of maps $\Xi_{\underline{x}}$, $\underline{x}$ spacelike, from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $\mathcal{H}_0$. These maps are obtained by composition of three linear maps which are defined as follows. The first one maps $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ onto $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O} + \underline{x})$ and is fixed by $\psi \times \psi' \longrightarrow \psi \times \psi'(\underline{x})$. Since the translations induce automorphisms of the field net this map is an isomorphism. The second one maps $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O} + \underline{x})$ into $\mathfrak{A}(\mathcal{O} \cup (\mathcal{O} + \underline{x}))$ and is given by

$$\psi \times \psi'(\underline{x}) \longrightarrow M(\psi \cdot \psi'(\underline{x})) \restriction \mathcal{H}_0 \ .$$

Since fields and observables commute at spacelike distances and since observables are invariant under the action of the mean M, it follows that the range of this map

is indeed contained in $\mathfrak{A}\big(\mathcal{O}\cup(\mathcal{O}+\underline{x})\big)$. Moreover, there holds $\|M(\sum \psi_i \cdot \psi_k'(\underline{x}))\| \leq \sum \|\psi_i\|\|\psi_k'(\underline{x})\|$, and since the left hand side of this inequality does not depend on the particular decomposition of the sum it is also clear that the map is bounded in norm by 1. The third map is $\Theta^{(0)}_{\beta,\mathcal{O}\cup(\mathcal{O}+\underline{x})}$, defined in Condition C and assumed to be compact. By composition of these maps we obtain the map $\Xi_{\underline{x}}$,

$$\Xi_{\underline{x}}(\psi \times \psi') = e^{-\beta H_0} M\big(\psi \cdot \psi'(\underline{x})\big)\Omega \ . \tag{3.12}$$

Since the ε-content of a compact map does not increase under composition with maps of norm less than or equal to 1, the following result is an immediate consequence of the preceding discussion.

Lemma 3.1. *Let Condition C be satisfied and let $\Xi_{\underline{x}}$, $\underline{x}$ spacelike, be the family of maps from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $\mathcal{H}_0$ defined above. Then each $\Xi_{\underline{x}}$ is compact and there holds in the limit of large spacelike translations $\underline{x}$*

$$\liminf_{\underline{x}} N_{\Xi_{\underline{x}}}(\varepsilon) \leq \underline{N}(\varepsilon) \ , \ \ \varepsilon > 0$$

where $\underline{N}(\varepsilon) \doteq \underline{N}_{\beta,\mathcal{O},\mathcal{O}}(\varepsilon)$, cf. Condition C.

In the next step we show that the preceding Lemma provides relevant information on the map Ξ_∞ from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $\mathcal{H} \otimes \mathcal{H}$, given by

$$\Xi_\infty(\psi \times \psi') = \int dg \big(V(g)e^{-\beta H}\psi\Omega\big) \otimes \big(V(g)e^{-\beta H}\psi'\Omega\big). \tag{3.13}$$

In the proof we make use of a weak form of the cluster theorem which does *not* depend on the existence of a mass gap in the theory.

Lemma 3.2. *Let Condition C be satisfied and let Ξ_∞ be the map from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $\mathcal{H} \times \mathcal{H}$ introduced above. Then Ξ_∞ is compact and*

$$N_{\Xi_\infty}(\varepsilon) \leq \underline{N}(\varepsilon) \ , \ \ \varepsilon > 0$$

with the same $\underline{N}(\varepsilon)$ as is the preceding Lemma.

Proof: The result is based on the fact that the maps $\Xi_{\underline{x}}$ dominate Ξ_∞ asymptotically in the sense of Lemma 2.4. To verify this statement let us consider for arbitrary $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{F}(\mathcal{O})$ the scalar product in $\mathcal{H}$

$$\big(\Xi_{\underline{x}}(\psi_1 \times \psi_1'), \Xi_{\underline{x}}(\psi_2 \times \psi_2')\big) = \int dg \big(\psi_1\psi_1'(\underline{x})\Omega, \ e^{-2\beta H} \ \psi_2^g \psi_2'^g(\underline{x})\Omega\big) \ . \tag{3.14}$$

Here we introduced the notation $\psi^g = V(g)\psi V(g)^{-1}$ and made use of the fact that (a) dg is the Haar measure on G and therefore invariant under left and right action of the group, (b) the gauge transformations $V(g)$ commute with translations and (c) the vacuum vector Ω is invariant under the action of $V(g)$. In order to calculate this scalar product in the limit of large spacelike $\underline{x}$ we make use of the fact that H is non–negative. Thus, by the spectral theorem, we can write

$$e^{-2\beta H} = \frac{1}{\sqrt{2\pi}} \int dt f_\beta(t) U(t) \ ,$$

where f_β is any test function whose Fourier transform $\tilde{f}_\beta$ satisfies $\tilde{f}_\beta(\omega) = e^{-2\beta\omega}$ for $\omega \geq 0$. Hence the scalar product in (3.14) can be represented in the form

$$\int dg \int dt f_\beta(t) \big(\psi_1 \psi_1'(\underline{x})\Omega \ , \ \psi_2^g(t)\psi_2'^g(t, \underline{x})\Omega\big)$$

since Ω is invariant under translations. We now make use of the spacelike (anti) commutation relations of the fields [24]: every field operator ψ can be decomposed into a sum of operators of Bose and Fermi type which commute, respectively anticommute at spacelike distances. Thus, without restriction of generality, we may assume that the operators ψ_1, ψ_1', ψ_2, ψ_2' are of fixed Bose or Fermi type. We treat here only the case where all operators are of Bose type, the other cases are analogous. Then, for fixed t and sufficiently large $\underline{x}$, there holds

$$\big(\psi_1\psi_1'(\underline{x})\Omega, \psi_2^g(t)\psi_2'^g(t, \underline{x})\Omega\big) = \big(\psi_2^g(t)^*\psi_1\Omega \ , \ U(\underline{x})\psi_1'^*\psi_2'^g(t)\Omega\big) \ ,$$

where we made again use of the fact that Ω is invariant under translations. The latter expression converges in the limit of large $\underline{x}$ to

$$\big(\psi_1\Omega \ , \ \psi_2^g(t)\Omega\big)\big(\psi_1'\Omega \ , \ \psi_2'^g(t)\Omega\big)$$

since, as a consequence of locality, the spectrum of the generators of spatial translations $U(\underline{x})$ is Lebesgue absolutely continuous, apart from a discrete part at 0 corresponding to the vacuum, cf. the arguments in [31, Sec. 2] which can easily be extended to Fermi fields. Hence, applying the dominated convergence theorem, we conclude that

$$\lim_{\underline{x}} \big(\Xi_{\underline{x}}(\psi_1 \times \psi_1') \, , \, \Xi_{\underline{x}}(\psi_2 \times \psi_2')\Omega\big) =$$
$$= \int dg \int dt \, f_\beta(t) \big(\psi_1\Omega \, , \, \psi_2^g(t)\Omega\big)\big(\psi_1'\Omega \, , \, \psi_2'^g(t)\Omega\big) =$$
$$= \int dg \big(\psi_1\Omega \, , \, e^{-2\beta H_0}\psi_2^g\Omega\big)\big(\psi_1'\Omega \, , \, e^{-2\beta H_0}\psi_2'^g\Omega\big) \ ,$$

where in the second equality we made again use of the positivity of H and the specific form of the function f_β. But the last term in this equation coincides with the scalar product $\big(\Xi_\infty(\psi_1 \times \psi_1')\,,\,\Xi_\infty(\psi_2 \times \psi_2')\big)$ in $\mathcal{H} \otimes \mathcal{H}$. Hence there holds

$$\lim_{\underline{x}} \|\Xi_{\underline{x}}(\sum \psi_i \times \psi_k')\| = \|\Xi_\infty(\sum \psi_i \times \psi_k')\|$$

for any $\sum \psi_i \times \psi_k' \in \mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$. It follows that the (uniformly continuous) family of maps $\Xi_{\underline{x}}$ dominates Ξ_∞ asymptotically, as claimed. The statement now follows from Lemma 2.4 and the preceding lemma. $\qquad\square$

In a final preparatory step we proceed from Ξ_∞ to a map Ξ from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ to the space $HS(\mathcal{H})$ of Hilbert Schmidt operators on $\mathcal{H}$. We recall that this space is equipped with the scalar product

$$(H_1, H_2)_{HS} \doteq Tr H_1^* H_2 \ , \quad H_1, H_2 \in HS(\mathcal{H}) \ . \tag{3.15}$$

The norm on $HS(\mathcal{H})$ will be denoted by $\|\ \ \|_{HS}$. We also recall that $HS(\mathcal{H})$ is canonically isomorphic to $\mathcal{H} \otimes \bar{\mathcal{H}}$, where $\bar{\mathcal{H}}$ is the conjugate space of $\mathcal{H}$. This isomorphism is established by assigning to the elements $\Phi_1 \otimes \bar{\Phi}_2 \in \mathcal{H} \otimes \bar{\mathcal{H}}$ the rank one operators $(\Phi_2, \cdot)\Phi_1 \in HS(\mathcal{H})$, and vice versa.

Let Ξ be the linear map from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $HS(\mathcal{H})$ given by

$$\Xi(\psi \times \psi') = \int dg\big(V(g)e^{-\beta H}\psi'^*\Omega, \cdot\big)V(g)e^{-\beta H}\psi\Omega \ . \tag{3.16}$$

The link between Ξ and Ξ_∞ is established with the help of Condition J. Making use of the antiunitary involution J and the fact that $J\mathcal{H}$ can be identified with $\bar{\mathcal{H}}$, we see that Ξ is isomorphic to the map from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $\mathcal{H} \otimes \bar{\mathcal{H}}$, given by

$$\psi \times \psi' \to \int dg\big(V(g)e^{-\beta H}\psi\Omega\big) \otimes \big(J\,V(g)e^{-\beta H}\psi'^*\Omega\big) \ . \tag{3.17}$$

Since J leaves Ω invariant and commutes with the gauge group as well as all real functions of H, the right hand side of this assignment can be rewritten in the form

$$\int dg\big(V(g)e^{-\beta H}\psi\Omega\big) \otimes \big(V(g)e^{-\beta H}J\psi'^*J\Omega\big) \ . \tag{3.18}$$

But with our choice of the region $\mathcal{O}$ there holds $J\mathcal{F}(\mathcal{O})J = \mathcal{F}(-\mathcal{O}) = \mathcal{F}(\mathcal{O})$, hence the map $\psi \times \psi' \to \psi \times J\psi'^*J$ defines an automorphism of the linear space $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$. Thus we conclude that the maps Ξ and Ξ_∞ are related by an isomorphism. The following result is then an immediate consequence.

Proposition 3.3. *Let Conditions C and J be satisfied and let Ξ be the map from the space $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $HS(\mathcal{H})$, defined above. Then Ξ is compact, and there holds*

$$N_\Xi(\varepsilon) \leq \underline{N}(\varepsilon) \ , \quad \varepsilon > 0 \ ,$$

with $\underline{N}(\varepsilon)$ as in Lemma 3.1.

4. Compactness properties of field algebras

The map Ξ, introduced in the preceding section, contains the desired information about the compactness properties of the map

$$\Theta \ \dot{=} \ \Theta_{\beta,\mathcal{O}} \tag{4.1}$$

as we shall demonstrate now. (Since $\mathcal{O}$ and β are kept fixed in the following we can simplify the notation and omit these subscripts.) Our strategy is as follows. We decompose $\mathcal{H}$ into the superselection sectors $\mathcal{H}_\iota^\sigma$, $\iota = 1, \cdots d_\sigma$ (as already indicated, each sector σ appears with multiplicity d_σ) and consider the corresponding maps $\Theta_\iota^\sigma = P_\iota^\sigma \cdot \Theta$, where P_ι^σ is the orthogonal projection onto $\mathcal{H}_\iota^\sigma$. As we will see, each of these maps is compact and has an ε–content which is controlled by that of Ξ. In order to obtain information on Θ we have to sum up the maps Θ_ι^σ, and it is here where Condition P enters. From that assumption it follows that also the map Θ is compact and that its ε–content is related to that of Ξ.

For any $\sigma \in \hat{G}$, corresponding to an irreducible unitary representation of G of dimension d_σ, we fix a unitary matrix representation $D_{\iota\kappa}^\sigma(\cdot)$, ι, $\kappa = 1, \cdots d_\sigma$. With the help of this representation we define the family of operators on $\mathcal{H}$ given by

$$P_\iota^\sigma \ \dot{=} \ \int dg D_{\iota\iota}^\sigma(g) V(g) \ . \tag{4.2}$$

It follows from the familiar orthogonality relations for unitary matrix representations of compact groups [32] that the operators P_ι^σ are orthogonal projections, $P_\iota^{\sigma*} P_{\iota'}^{\sigma'} = \delta_{\sigma\sigma'} \delta_{\iota\iota'} P_\iota^\sigma$. Moreover, each vector in $\mathcal{H}_\iota^\sigma \dot{=} P_\iota^\sigma \mathcal{H}$ transforms under gauge transformations $V(g)$ according to the irreducible representation σ of G. The orbits of these vectors span the space $\mathcal{H}^\sigma \dot{=} P^\sigma \mathcal{H}$, $P^\sigma \dot{=} \sum_{\iota=1}^{d_\sigma} P_\iota^\sigma$, i.e., the unique subspace of $\mathcal{H}$ on which V acts like a multiple of the representation σ and which is stable under the action of the fixed point net $\mathcal{F}^G$. Accordingly, there holds the completeness relation $\sum_{\sigma \in \hat{G}} P^\sigma = 1$, where the limit of infinite sums is understood in the strong operator topology.

Next, we define maps M_ι^σ from $\mathcal{F}(\mathcal{O})$ into $\mathcal{F}(\mathcal{O})$, setting

$$M_\iota^\sigma(\psi) \doteq \int dg D_{\iota\iota}^\sigma(g^{-1}) V(g) \psi V(g)^{-1} \ . \tag{4.3}$$

Since the matrices $D_{\iota\kappa}^\sigma(g^{-1})$ are unitary, there holds $|D_{\iota\kappa}^\sigma(g^{-1})| \leq 1$ and consequently each map M_ι^σ is bounded in norm by 1. Hence the linear map from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into itself, which is given by

$$\psi \times \psi' \longrightarrow M_\iota^\sigma(\psi) \times M_\iota^\sigma(\psi'), \tag{4.4}$$

also has norm less than or equal to 1. We compose the latter map with Ξ and thereby obtain a map Ξ_ι^σ from $\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})$ into $HS(\mathcal{H})$. This map clearly has the same compactness properties as those stated for Ξ in Proposition 3.3.

Making use of this fact we want to exhibit compactness properties of the maps

$$\Theta_\iota^\sigma \doteq P_\iota^\sigma \cdot \Theta \ . \tag{4.5}$$

In order to simplify the notation we keep σ, ι fixed for a moment and put, for given $\psi \in \mathcal{F}(\mathcal{O})$,

$$\Psi \doteq P_\iota^\sigma e^{-\beta H} \psi \Omega \ . \tag{4.6}$$

The orthogonal projection onto the ray of Ψ is denoted by E_Ψ. It will be crucial in the following that the orbit of any such vector Ψ under the action of the gauge transformations $V(g)$ gives rise to an irreducible representation of G.

With the above notation we can write

$$\Xi_\iota^\sigma(\psi \times \psi^*) = \int dg \big(V(g)\Psi, \cdot\big) V(g)\Psi = \|\Psi\|^2 \cdot M(E_\Psi) \ . \tag{4.7}$$

Hence, for any pair of operators $\psi, \psi' \in \mathcal{F}(\mathcal{O})$, there holds

$$\big\|\Xi_\iota^\sigma(\psi \times \psi^* - \psi' \times \psi'^*)\big\|_{HS} = \big\|\|\Psi\|^2 M(E_\Psi) - \|\Psi'\|^2 M(E_{\Psi'})\big\|_{HS} \ . \tag{4.8}$$

An important lower bound for the right hand side of this inequality is given in the following lemma.

Lemma 4.1. *Let* $\Psi, \Psi' \in \mathcal{H}_\iota^\sigma$*. Then*

$$\big\|\|\Psi\|^2 M(E_\Psi) - \|\Psi'\|^2 M(E_{\Psi'})\big\|_{HS} \geq \frac{1}{\sqrt{2d_\sigma}} \inf_{\bar{V}} \|\Psi - \bar{V}\Psi'\|^2 \ ,$$

where the infimum is understood with respect to all unitary operators

$$\bar{V} \in \left\{ V(g) \upharpoonright \mathcal{H}^{\sigma} : g \in G \right\}'' .$$

Remark: Since the restriction of the representation V of G to the subspace $\mathcal{H}^{\sigma}$ is equivalent to a multiple of the irreducible representation σ it follows that the group generated by the unitaries $\bar{V}$ in the statement is isomorphic to the group $\mathcal{U}(d_{\sigma})$ of all unitaries on a d_{σ}–dimensional Hilbert space.

Proof: Since the operator $M(E_{\Psi})$ commutes with $V(g)$, $g \in G$, it follows from Schur's Lemma that it is a multiple of the projection operator I_{Ψ} which projects onto the d_{σ}–dimensional irreducible subspace of the representation V, containing Ψ. Hence, by computing traces, one finds that $M(E_{\Psi}) = d_{\sigma}^{-1} \cdot I_{\Psi}$. An analogous statement holds for Ψ'. Bearing in mind the definition of the scalar product in $HS(\mathcal{H})$ it follows that

$$\begin{aligned}
\left\| \|\Psi\|^2 M(E_{\Psi}) - \|\Psi'\|^2 M(E_{\Psi'}) \right\|_{HS}^2 &= \\
&= d_{\sigma}^{-1} \|\Psi\|^4 + d_{\sigma}^{-1} \|\Psi'\|^4 - 2 d_{\sigma}^{-2} \|\Psi\|^2 \|\Psi'\|^2 Tr I_{\Psi} I_{\Psi'} \\
&\geq d_{\sigma}^{-1} \left(\|\Psi\|^4 + \|\Psi'\|^4 - 2\|\Psi\|^2 \|\Psi'\|^2 \|I_{\Psi} I_{\Psi'}\|^2 \right) \\
&= d_{\sigma}^{-1} \left(\|\Psi\|^4 + \|\Psi'\|^4 - 2 \sup_{\bar{V}} |(\Psi, \bar{V}\Psi')|^2 \right) .
\end{aligned}$$

Here we made use of the fact (in the inequality) that $|Tr I_{\Psi} I_{\Psi'}| = Tr I_{\Psi} I_{\Psi'} I_{\Psi} \leq d_{\sigma} \cdot \|I_{\Psi} I_{\Psi'}\|^2$ and (in the last equality) that the unitaries $\bar{V} \in \{V(g) \upharpoonright \mathcal{H}^{\sigma} : g \in G\}''$ act transitively on the unit ball of every irreducible subspace of the representation V in $\mathcal{H}^{\sigma}$. By an elementary calculation one sees that for $Re(\Psi, \bar{V}\Psi') \geq 0$

$$\begin{aligned}
\frac{1}{2} \|\Psi - \bar{V}\Psi'\|^4 &= \frac{1}{2} \left(\|\Psi\|^2 + \|\Psi'\|^2 - 2Re(\Psi, \bar{V}\Psi') \right)^2 \\
&\leq \|\Psi\|^4 + \|\Psi'\|^4 - 2 \left(Re(\Psi, \bar{V}\Psi') \right)^2 .
\end{aligned}$$

Hence, since the set of unitaries $\bar{V}$ is stable under multiplication with phase factors $\eta \in \mathbb{C}$, $|\eta| = 1$, there holds

$$\frac{1}{2} \inf_{\bar{V}} \|\Psi - \bar{V}\Psi'\|^2 \leq \|\Psi\|^4 + \|\Psi'\|^4 - 2 \sup_{\bar{V}} |(\Psi, \bar{V}\Psi')|^2 .$$

The statement then follows. $\qquad \square$

Making use of this lemma and recalling that $\Psi = P_{\iota}^{\sigma} e^{-\beta H} \psi \Omega = \Theta_{\iota}^{\sigma}(\psi)$, we can proceed from (4.8) to the estimate

$$\sqrt{2 d_{\sigma}} \, \|\Xi_{\iota}^{\sigma}(\psi \times \psi^* - \psi' \times \psi'^*)\|_{HS} \geq \inf_{\bar{V}} \|\Theta_{\iota}^{\sigma}(\psi) - \bar{V} \cdot \Theta_{\iota}^{\sigma}(\psi')\|^2 \qquad (4.9)$$

which holds for any $\psi, \psi' \in \mathcal{F}(\mathcal{O})$. We are now in the position to establish the following result.

Proposition 4.2. *Let Conditions C and J be satisfied and let Θ_ι^σ be the map from $\mathcal{F}(\mathcal{O})$ into $\mathcal{H}_\iota^\sigma$, defined in (4.5). Then Θ_ι^σ is compact, and its ε–content is bounded by*

$$N_{\Theta_\iota^\sigma}(\varepsilon) \leq \inf_{\varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2}} \underline{N}\left(\frac{\varepsilon_1^2}{\sqrt{2d_\sigma}}\right) \cdot \left(\frac{2d_\sigma e^{-\beta m_\sigma}}{\varepsilon_2} + 1\right)^{d_\sigma^2}, \varepsilon > 0 ,$$

where m_σ is the lower boundary of the mass spectrum in the sector σ and $\underline{N}(\varepsilon)$ is defined in Lemma 3.1.

Proof: For the proof of the statement we make use of the estimate (4.9) as well as of Lemma 2.6 and 2.7. Let us assume that there are, for given $\varepsilon > 0$, $M(\varepsilon)$ elements $\psi_i \in \mathcal{F}(\mathcal{O})_1$, $i = 1, \cdots M(\varepsilon)$, such that $\inf_{\bar{V}} \|\Theta_\iota^\sigma(\psi_i) - \bar{V}\Theta_\iota^\sigma(\psi_k)\| > \varepsilon$ for $i \neq k$. It then follows from (4.9) that

$$\|\Xi_\iota^\sigma(\psi_i \times \psi_i^* - \psi_k \times \psi_k^*)\|_{HS} > \frac{\varepsilon^2}{\sqrt{2d_\sigma}} , \quad i \neq k .$$

But this implies, since $\psi_i \times \psi_i^* \in \left(\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{O})\right)_1$, $i = 1, \cdots M(\varepsilon)$, that $M(\varepsilon)$ cannot be larger than the $\varepsilon^2/\sqrt{2d_\sigma}$–content of Ξ_ι^σ, $M(\varepsilon) \leq N_{\Xi_\iota^\sigma}\left(\frac{\varepsilon^2}{\sqrt{2d_\sigma}}\right)$. Next we recall that the set of unitaries $\bar{V} \in \left\{V(g) \restriction \mathcal{H}^\sigma : g \in G\right\}''$ forms a group $\bar{\mathcal{V}}$ which is isomorphic to the group of unitaries $\mathcal{U}(d_\sigma)$. Thus it follows from Lemma 2.6 that the ε–content of $\bar{\mathcal{V}}$ is bounded by

$$N^{\bar{\mathcal{V}}}(\varepsilon) \leq \left(\frac{2d_\sigma}{\varepsilon} + 1\right)^{d_\sigma^2} , \quad \varepsilon > 0 .$$

We also note that $\|\Theta_\iota^\sigma(\psi)\| = \|P_\iota^\sigma e^{-\beta H}\psi\Omega\| \leq e^{-\beta m_\sigma}\|\psi\|$ and consequently $\|\Theta_\iota^\sigma\| \leq e^{-\beta m_\sigma}$. We can now apply Lemma 2.7 to the set $\mathcal{X} = \Theta_\iota^\sigma\left(\mathcal{F}(\mathcal{O})_1\right)$, giving

$$N_{\Theta_\iota^\sigma}(\varepsilon) \leq \inf_{\varepsilon_1 + e^{-\beta m_\sigma}\varepsilon_2 = \varepsilon/2} M(\varepsilon_1) \cdot N^{\bar{\mathcal{V}}}(\varepsilon_2) .$$

Hence the statement follows from the preceding estimates, Proposition 3.3, and the fact that the ε–content of Ξ_ι^σ is not larger than that of Ξ. $\qquad \square$

This result provides information on the compactness properties of the components of the map Θ in the various superselection sectors of the theory. Since, for $\psi \in \mathcal{F}(\mathcal{O})$,

$$\sum_{\sigma \in \hat{G}} \sum_{\iota=1}^{d_\sigma} \Theta_\iota^\sigma(\psi) = \left(\sum_{\sigma \in \hat{G}} \sum_{\iota=1}^{d_\sigma} P_\iota^\sigma\right) \cdot \Theta(\psi) = \Theta(\psi) \tag{4.10}$$

one obtains with the help of Lemma 2.3 a similar result for the map Θ if there are only a finite number of superselection sectors in the theory. In order to obtain information on Θ in the general case we make use of Condition P. As was discussed in the proof of the preceding proposition, there holds $\|\Theta_\iota^\sigma\| \leq e^{-\beta m_\sigma}$, and consequently

$$\sum_{\sigma \in \hat{G}} \sum_{\iota=1}^{d_\sigma} \|\Theta_\iota^\sigma\| \leq \sum_{\sigma \in \hat{G}} d_\sigma e^{-\beta m_\sigma} = \mathcal{Z}(\beta) < \infty \tag{4.11}$$

if Condition P is satisfied. Thus in this case Θ is equal to an absolutely converging sum of compact maps and therefore it is also compact. (As a matter of fact, this result follows from considerably weaker versions of Condition P. It would suffice for example if, for $\varepsilon > 0$, $e^{-\beta m_\sigma} < \varepsilon$ for almost all superselection sectors σ.)

It is not quite as easy to estimate the ε–content of Θ. In order to abreviate the argument we do not aim here at an optimal estimate and proceed as follows. We pick $\alpha > 0$ and consider the maps $e^{-\alpha H} \cdot \Theta_\iota^\sigma$. Since $\|e^{-\alpha H} \cdot \Theta_\iota^\sigma(\psi)\| \leq e^{-\alpha m_\sigma}\|\Theta_\iota^\sigma(\psi)\|$ we obtain for the respective ε–contents of these maps the inequality $N_{e^{-\alpha H} \cdot \Theta_\iota^\sigma}(\varepsilon) \leq N_{\Theta_\iota^\sigma}(e^{\alpha m_\sigma} \cdot \varepsilon)$, $\varepsilon > 0$. Next we consider, for fixed σ, the map $e^{-\alpha H} \cdot \Theta^\sigma \doteq \sum_{\iota=1}^{d_\sigma} e^{-\alpha H} \cdot \Theta_\iota^\sigma$. It follows from Lemma 2.3, the preceding bounds, and Proposition 4.2 that, for $\varepsilon > 0$,

$$N_{e^{-\alpha H} \cdot \Theta^\sigma}(\varepsilon) \leq \prod_{\iota=1}^{d_\sigma} N_{\Theta_\iota^\sigma}\left(\frac{e^{\alpha m_\sigma}}{2d_\sigma}\varepsilon\right)$$

$$\leq \inf_{\varepsilon_1 + \varepsilon_1' = \frac{\varepsilon}{2}} N\left(\frac{e^{2\alpha m_\sigma}}{(2d_\sigma)^{5/2}}\varepsilon_1^2\right)^{d_\sigma} \left(\frac{(2d_\sigma)^2 e^{-(\alpha+\beta)m_\sigma}}{\varepsilon_1'} + 1\right)^{d_\sigma^3} . \tag{4.12}$$

Finally, we consider for finite subsets $\hat{F} \subset \hat{G}$ the maps $e^{-\alpha H} \cdot \Theta^{\hat{F}} = \sum_{\sigma \in \hat{F}} e^{-\alpha H} \cdot \Theta^\sigma$. Since for $\psi \in \mathcal{F}(\mathcal{O})$

$$e^{-\alpha H} \cdot \Theta^{\hat{F}}(\psi) = \sum_{\sigma \in \hat{F}} \sum_{\iota=1}^{d_\sigma} P_\iota^\sigma \cdot e^{-(\alpha+\beta)H}\psi\Omega \tag{4.13}$$

it follows that $e^{-\alpha H} \cdot \Theta^{\hat{F}}(\psi)$ converges strongly to $e^{-\alpha H} \cdot \Theta(\psi)$ for any increasing net $\hat{F} \nearrow \hat{G}$. Hence, for any such net, the family of maps $e^{-\alpha H} \cdot \Theta^{\hat{F}}$ dominates $e^{-\alpha H} \cdot \Theta$ asymptotically in the sense of Lemma 2.4. So in order to get a bound for the ε–content of $e^{-\alpha H} \cdot \Theta$ we have to establish uniform bounds for the ε–contents of the maps $e^{-\alpha H} \cdot \Theta^{\hat{F}}$. To this end we make again use of Lemma 2.3, from which it follows that

$$N_{e^{-\alpha H} \cdot \Theta^{\hat{F}}}(\varepsilon) \leq \inf_{\varepsilon_1 + \cdots + \varepsilon_F = \frac{\varepsilon}{2}} N_{e^{-\alpha H} \cdot \Theta^{\sigma_1}}(\varepsilon_1) \cdots N_{e^{-\alpha H} \cdot \Theta^{\sigma_F}}(\varepsilon_F) , \tag{4.14}$$

where F is the cardinality of $\hat{F}$ and $\sigma_i \in \hat{F}$, $i = 1, \cdots F$. Plugging into this estimate the preceding bounds on $N_{e^{-\alpha H} \cdot \Theta^\sigma}$ we get

$$N_{e^{-\alpha H} \cdot \Theta^{\hat{F}}}(\varepsilon) \leq \inf_{\varepsilon_1 + \cdots + \varepsilon_F + \varepsilon_1' + \cdots + \varepsilon_F' = \frac{\varepsilon}{2}} \prod_{i=1}^{F} N\left(\frac{e^{2\alpha m_{\sigma_i}}}{(2d_{\sigma_i})^{5/2}} \varepsilon_1^2 \right)^{d_{\sigma_i}} \times$$

$$\times \prod_{j=1}^{F} \left(\frac{(2d_{\sigma_j})^2 e^{(\alpha+\beta)m_{\sigma_j}}}{\varepsilon_j'} + 1 \right)^{d_{\sigma_j}^3} . \tag{4.15}$$

We proceed to an upper bound of the right hand side of this inequality by restricting the infimum to the partitions $\varepsilon_1 + \cdots + \varepsilon_F = \varepsilon_1' + \cdots + \varepsilon_F' = \frac{\varepsilon}{4}$. This allows us to treat separately the "kinematical factor" in this expression.

Lemma 4.3. *Let $0 < r \leq 1$. There holds for $\varepsilon > 0$*

$$\inf_{\varepsilon_1' + \cdots \varepsilon_F' = \frac{\varepsilon}{4}} \prod_{j=1}^{F} \left(\frac{(2d_{\sigma_j})^2 e^{-(\alpha+\beta)m_{\sigma_j}}}{\varepsilon_j'} + 1 \right)^{d_{\sigma_j}^3} \leq \exp\left(\frac{1}{r} \mathcal{Z} \left(\frac{r(\alpha+\beta)}{3+2r} \right)^{3+2r} \left(\frac{16}{\varepsilon} \right)^r \right)$$

uniformly for all finite subsets $\hat{F} \subset \hat{G}$.

Proof: Since for fixed $r > 0$ and any $\lambda \geq 0$ there holds $ln(1 + \lambda) \leq r^{-1}\lambda^r$ one gets for the left hand side of the inequality in the statement the upper bound

$$\inf_{\varepsilon_1' + \cdots + \varepsilon_F' = \frac{\varepsilon}{4}} \exp\left(\frac{1}{r} \sum_{j=1}^{F} d_{\sigma_j}^3 \frac{(2d_{\sigma_j})^{2r} e^{-r(\alpha+\beta)m_{\sigma_j}}}{\varepsilon_j'^r} \right) .$$

This infimum can be computed by an application of the Lagrange multiplier method. It is equal to

$$\exp\left(\frac{1}{r} \left(\frac{16}{\varepsilon} \right)^r \left(\sum_{j=1}^{F} d_{\sigma_j}^{(3+2r)/(1+r)} e^{-r(\alpha+\beta)m_{\sigma_j}/(1+r)} \right)^{(1+r)} \right) .$$

Since $(3+2r)/(1+r) \geq 1$ the sum in this expression can be estimated by

$$\left(\sum_{j=1}^{F} d_{\sigma_j}^{(3+2r)/(1+r)} e^{-r(\alpha+\beta)m_{\sigma_j}/(1+r)} \right)^{(1+r)}$$

$$\left(\sum_{j=1}^{F} d_{\sigma_j} e^{-r(\alpha+\beta)m_{\sigma_j}/(3+2r)} \right)^{(3+2r)} \leq \mathcal{Z} \left(\frac{r(\alpha+\beta)}{(3+2r)} \right)^{3+2r} .$$

Hence the statement follows. □

Thus the kinematical factor in our estimate of the ε–content of $e^{-\alpha H} \cdot \Theta^{\hat{F}}$ will in general develop in the limit $\hat{F} \nearrow \hat{G}$ an essential singularity as $\varepsilon \searrow 0$. But it follows from our result that this singularity is only of order 0 if Condition P is satisfied.

Let us discuss now the properties of the term involving the factors $\underline{N}(\varepsilon)$. Since $\underline{N}(\varepsilon)$ is the *limes inferior* of the ε–content of maps with norm less than or equal to 1, there holds $\underline{N}(\varepsilon) = 1$ if $\varepsilon > 2$. It therefore follows from Condition P that

$$\sup_{\hat{F} \subset \hat{G}} \inf_{\varepsilon_1 + \cdots + \varepsilon_F = \frac{\varepsilon}{4}} \prod_{i=1}^{F} \underline{N}\left(\frac{e^{2\alpha m_{\sigma_i}}}{(2d_{\sigma_i})^{5/2}} \varepsilon_i^2 \right)^{d_{\sigma_i}} < \infty . \tag{4.16}$$

In order to verify this statement we note that the above infimum increases if we proceed to the condition $\varepsilon_1 + \cdots + \varepsilon_F \leq \varepsilon/4$ since $\underline{N}(\varepsilon)$ increases if ε decreases. Setting

$$\varepsilon_i = \mathcal{Z} \left(\frac{2}{5}\alpha \right)^{-5/4} d_{\sigma_i}^{5/4} e^{-\alpha m_{\sigma_i}/2} \cdot \frac{\varepsilon}{4}, \quad i = 1, \cdots F \tag{4.17}$$

there holds $(\varepsilon_1 + \cdots + \varepsilon_F) \leq \frac{\varepsilon}{4}$. Hence the left hand side of (4.16) is dominated by

$$\sup_{\hat{F} \subset \hat{G}} \prod_{i=1}^{F} \underline{N}\left(2^{-13/2} Z \left(\frac{2}{5}\alpha \right)^{-5/2} e^{\alpha m_{\sigma_i} \varepsilon^2} \right)^{d_{\sigma_i}} . \tag{4.18}$$

Now $e^{\alpha m_{\sigma_i}}$ is, for $\alpha > 0$, larger than any given constant on almost all superselection sectors σ if Condition P is satisfied. Hence, for any $\varepsilon > 0$, only a finite number of factors in the above products are different from 1, so the supremum exists. Consequently we get, for $r > 0$ and $\varepsilon > 0$,

$$N_{e^{-\alpha H} \cdot \Theta}(\varepsilon) \leq \exp\left(\frac{1}{r} \mathcal{Z} \left(\frac{r(\alpha+\beta)}{3+2r} \right)^{3+2r} \left(\frac{16}{\varepsilon} \right)^r \right) \times$$

$$\times \prod_{\sigma \in \Sigma} \underline{N}\left(2^{-13/2} \mathcal{Z} \left(\frac{2}{5}\alpha \right)^{-5/2} e^{\alpha m_\sigma \varepsilon^2} \right)^{d_\sigma} . \tag{4.19}$$

It remains to proceed from this bound for the ε–content of the map $e^{-\alpha H} \cdot \Theta$ to a corresponding bound for the map Θ. This is accomplished with the help of the following result whose proof is taken from [3].

Lemma 4.4. *Let $\alpha > 0$. Then*

$$\|\Theta(\psi)\| \leq \|e^{-\alpha H} \cdot \Theta(\psi)\|^{\beta/(\alpha+\beta)} \cdot \|\psi\|^{\alpha/(\alpha+\beta)}, \quad \psi \in \mathcal{F}(\mathcal{O}) .$$

Proof: Making use of the spectral theorem, we can write

$$\|\Theta(\psi)\|^2 = (\psi\Omega, e^{-2\beta H}\psi\Omega) = \int e^{-2\beta\omega}d\mu(\omega) \ ,$$

where μ is a measure with support on $\mathbb{R}_+$. Because of Hölder's inequality and the fact that $e^{-\lambda\omega} \leq 1$ for $\lambda, \omega \geq 0$ there holds

$$\int e^{-2\beta\omega}d\mu(\omega) \leq \left(\int e^{-2(\alpha+\beta)\omega}d\mu(\omega)\right)^{\beta/(\alpha+\beta)}\left(\int d\mu(\omega)\right)^{\alpha/(\alpha+\beta)} \ ,$$

and since $\int d\mu(\omega) = \|\psi\Omega\|^2 \leq \|\psi\|^2$, the statement follows. $\qquad\square$

We can now state our main result.

Theorem 4.5. *Let Conditions C, J and P be satisfied. Then the maps $\Theta_{\beta,\mathcal{O}}$, defined in (3.9), are compact. Their ε–contents are, for any $0 < \alpha \leq \beta$, $0 < r \leq 1$, bounded by*

$$N_{\Theta_{\beta,\mathcal{O}}}(\varepsilon) \leq exp\ \left(\frac{2^5}{r}\mathcal{Z}\left(\frac{r}{5}\beta\right)^5 \left(\frac{1}{\varepsilon}\right)^{r(1+\alpha/\beta)}\right) \times$$

$$\times \prod_{\sigma\in\hat{G}} \underline{N}\left(2^{-17/2}\mathcal{Z}\left(\frac{2}{5}\alpha\right)^{-5/2} e^{\alpha m_\sigma}\varepsilon^{2(1+\alpha/\beta)}\right)^{d_\sigma} \ , \quad \varepsilon > 0.$$

Here $\mathcal{Z}$ is the little partition function, defined in Condition P, and $\underline{N}(\varepsilon)$ is the limes inferior of the ε–contents of the maps $\Theta^{(0)}_{\beta,\mathcal{O}\cup(\mathcal{O}+\underline{x})}$, defined in Condition C, for $\underline{x}$ tending to spacelike infinity.

Remark: The optimal choice of the parameters α, r depends on the detailed properties of $\mathcal{Z}$ and $\underline{N}$.

Proof: According to Lemma 4.4 the ε–content of Θ is not larger than the $2^{-\alpha/\beta}\varepsilon^{(1+\alpha/\beta)}$–content of the map $e^{-\alpha H} \cdot \Theta$. Making use of the estimate (4.19) on the ε–content of the latter map and the fact that the little partition function $\mathcal{Z}(\lambda)$ is monotonically increasing if λ decreases, the statement follows. $\qquad\square$

The preceding theorem makes clear how the compactness properties of the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ determine the compactness properties of their extensions $\Theta_{\beta,\mathcal{O}}$ to the local field algebras. It gives the desired information on the general state of affairs, but in applications one is frequently interested in a more explicit description of the compactness properties of the respective maps. We therefore introduce the following quantitative version of Condition C.

Condition N_p: Let $p > 0$, $\beta > 0$ be fixed and let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$. The maps $\Theta^{(0)}_{\beta, \mathcal{O}_1 \cup \mathcal{O}_2}$, defined in Condition C, are p–nuclear (for the given p) and

$$\liminf_{\underline{x}} \|\Theta^{(0)}_{\beta, \mathcal{O}_1 \cup (\mathcal{O}_2 + \underline{x})}\|_p < \infty$$

for $\underline{x}$ tending to spacelike infinity.

Making use of this condition we can establish the following interesting corollary.

Corollary 4.6. *Let Conditions P, J and N_p be satisfied for some $0 < p < \frac{1}{4}$. Then the maps $\Theta_{\beta, \mathcal{O}}$, defined in (3.9), are q–nuclear for $q > \frac{2p}{1-2p}$. Moreover, if $\frac{2p}{1-2p} < q \leq 4p$ there holds*

$$\|\Theta_{\beta, \mathcal{O}}\|_q \leq c_{p,q} \cdot \mathcal{Z}\left(\frac{1}{15}\left(q - \frac{2p}{1 - 2p}\right)\beta\right)^{5/2p} \cdot \liminf_{\underline{x}} \|\Theta^{(0)}_{\beta, \mathcal{O} \cup (\mathcal{O} + \underline{x})}\|_p^{1/2},$$

where $c_{p,q}$ is some numerical constant.

Proof: We sketch the argument which is a simple consequence of Proposition 2.5 and the preceding theorem. One first notices that by Condition N_p, the third part of Proposition 2.5 and relation (2.3) it follows that for $p' > p/(1 - p)$

$$\underline{N}(\varepsilon) = \liminf_{\underline{x}} N_{\Theta^{(0)}_{\beta, \mathcal{O} \cup (\mathcal{O} + \underline{x})}}(\varepsilon) \leq e^{(const \cdot L/\varepsilon)^{p'}},$$

where here and in the following *const* stands for numerical constants, depending only on the parameters p, p', etc., and $L \doteq \liminf_{\underline{x}} \|\Theta^{(0)}_{\beta, \mathcal{O} \cup (\mathcal{O} + \underline{x})}\|_p$. Plugging this information into the statement of the theorem and putting there $r = 2p'$ one finds that for $p/(1 - p) < p' \leq 1/2$ and $0 < \alpha \leq \beta$ there holds

$$M_{\Theta_{\beta, \mathcal{O}}}\left(2p'(1 + \alpha/\beta)\right) \leq const \; \mathcal{Z}\left(\frac{2}{5}p\alpha\right)^{5/2p} \cdot L^{1/2}.$$

Here the monotonicity properties of $\mathcal{Z}(\lambda)$ and the fact that $\mathcal{Z}(\lambda) \geq 1$ and $L \geq 1$ have been used in order to simplify the expression. By applying to this estimate the first two parts of Proposition 2.5 one arrives at

$$\|\Theta_{\beta, \mathcal{O}}\|_q \leq const \; \mathcal{Z}\left(\frac{2}{5}p\alpha\right)^{5/2p} \cdot L^{1/2}$$

for $1 \geq q > 2p(1 + \alpha/\beta)/\left(1 - p(2 + \alpha/\beta)\right)$. Since α can be made arbitrarily small, the first part of the statement follows. The quantitative estimate is obtained if one puts $\alpha/\beta = \frac{1}{6p}\left(q - \frac{2p}{1-2p}\right)$. $\qquad \square$

5. Applications

Starting from general, physically motivated assumptions we have established a tight relation between phase space properties of observables and charged fields in relativistic quantum field theory. These properties are encoded in specific features of the maps $\Theta^{(0)}_{\beta,\mathcal{O}\cup(\mathcal{O}+\underline{x})}$ which can be described in qualitative and quantitative terms by compactness or nuclearity conditions. We indicate here some applications of these results to problems of physical interest and mention some open questions.

One of the first applications of compactness respectively nuclearity conditions has been the discussion in [1] of the problem of causal (statistical) independence in relativistic quantum field theory. On the mathematical side this problem amounts to the question of whether the underlying net of observables or fields has the so-called *split property* [18]. We recall that a local net is said to have the split property if for each pair of double cones $\mathcal{O}_1$, $\mathcal{O}_2$ such that the closure of $\mathcal{O}_1$ is contained in the interior of $\mathcal{O}_2$ there exists a factor $\mathcal{M}$ of type I_∞ (i.e., a von Neumann algebra which is isomorphic to the algebra of all bounded operators on $\mathcal{H}$) such that $\mathfrak{A}(\mathcal{O}_1) \subset \mathcal{M} \subset \mathfrak{A}(\mathcal{O}_2)$, and similarly for the field algebras. It may happen that, for given $\mathcal{O}_1$, such factors $\mathcal{M}$ exist only for sufficiently large $\mathcal{O}_2$. The net is then said to have the *distal split property*.

It has been demonstrated in [1-5], cf. also [33], that the (distal) split property of local nets is closely related to nuclearity properties of the associated maps $\Theta^{(0)}_{\beta,\mathcal{O}}$, respectively $\Theta_{\beta,\mathcal{O}}$. The present results show that nuclearity properties of the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$, involving the observables, imply that the field net has the split property. We state this fact, which is a straightforward consequence of Corollary 4.6, Proposition 2.5 and the results in reference [5] in form of a theorem.

Theorem 5.1. *Let Condition P, J and N_p be satisfied for some $0 < p < \frac{1}{10}$. Then the field net $\mathcal{O} \to \mathcal{F}(\mathcal{O})$ has the distal split property. If condition N_p is satisfied for all $p > 0$ the field net has the split property.*

We note that this theorem has a partial converse. Namely, if the field net has the distal split property, then the maps $\Theta_{\beta,\mathcal{O}}$ are compact [3]. It is also noteworthy that the split property of the field net is a direct consequence of the split property of the observables if the gauge group is finite abelian [18]. The split property of the field net has several interesting consequences. We mention here only the fact that the existence of local generators for internal, geometrical and supersymmetry transformations can be established if this property holds [19,20]. One thereby arrives at a rigorous version of "current algebra".

Another field of applications of compactness and nuclearity conditions is the structural analysis of thermal states in relativistic quantum field theory. In these

applications one frequently has to know how the nuclear p-norms of the relevant maps depend on the size of the region $\mathcal{O}$ and the value of β. According to the heuristic considerations in [1,3], the p-norms may be interpreted as partition functions. Hence, anticipating decent thermal properties, it seems reasonable to expect (and can be established in models) that

$$\liminf_{\underline{x}} \|\Theta^{(0)}_{\beta,\mathcal{O}\cup(\mathcal{O}+\underline{x})}\|_p \leq e^{cr^m\beta^{-n}} \tag{5.1}$$

for sufficiently large r, β^{-1}, where r is the diameter of $\mathcal{O}$ and the (positive) constants c,m,n may depend on p. Similarly, there should hold for the little partition function

$$\mathcal{Z}(\lambda) \leq e^{c\lambda^{-l}} \tag{5.2}$$

for small $\lambda > 0$ and positive constants c and l. It then follows from Corollary 4.6 that a similar bound as in (5.1) holds for the norms $\|\Theta_{\beta,\mathcal{O}}\|_q$, $q > \frac{2p}{1-2p}$. One can therefore apply the results in [6] and establish the existence of thermal equilibrium states for the smooth subnet[1] $\mathcal{F}_0$ of the field net $\mathcal{F}$. We recall that thermal equilibrium states are distinguished by the fact that they satisfy the KMS-condition with respect to time translations [34].

Theorem 5.2. *Let Condition J and the specific versions* (5.1) *and* (5.2) *of Conditions N_p and P be satisfied for some $0 < p < \frac{1}{4}$. Then there exist KMS-states for the (smooth) field net $\mathcal{F}_0$ for all positive temperatures $\beta^{-1} > 0$.*

Appealing to the thermodynamical interpretation of the nuclearity conditions [1,3], this result can be rephrased in more physical terms: if the partition function of the canonical ensemble with zero total charge exists and exhibits the physically expected behaviour (relation (5.1)) and if the mass spectrum of the theory is sufficiently tame (relation (5.2)) then the grand canonical ensemble with zero chemical potential exists in the thermodynamical limit.

By a straightforward generalization of Condition P this result can be extended to the grand canonical ensembles with non-zero chemical potential. To illustrate this fact let us assume that Q is the generator of a one-parameter subgroup of the gauge transformations with eigenvalue q_σ on the superselection sectors σ and let $\mu \in \mathbb{R}$ be such that the function $\mathcal{Z}_\mu(\lambda) = \sum_\sigma d_\sigma e^{-\lambda(m_\sigma + \mu q_\sigma)}$ satisfies condition (5.2). (This would be implied for sufficiently small μ by relation (5.2) if the respective charge is

---

[1]The smooth subnet $\mathcal{F}_0$ of $\mathcal{F}$ is generated by all operators $\psi \in \mathcal{F}$ which transform norm continuously under time translations. It is dense in $\mathcal{F}$ in the strong operator topology.

tied to massive particles.) It then follows from the arguments given in Sec. 4 that the maps

$$\Theta_{\beta,\mu,\mathcal{O}}(\psi) = e^{-\beta(H+\mu Q)}\psi\Omega, \quad \psi \in \mathcal{F}(\mathcal{O}) , \tag{5.3}$$

are q-nuclear and that conditions (5.1) and (5.2) lead to corresponding upper bounds for their respective q-norms, $q > \frac{2p}{1-2p}$. Hence, by the results in [6], there exist KMS-states for the (smooth) field net and the dynamics $U_\mu(t) = e^{it(H+\mu Q)}$, i.e., states with chemical potential μ [35].

It is a remarkable fact that in theories of local observables the existence of the grand canonical ensembles is implied by the existence of the (neutral) canonical ensemble and purely kinematical conditions. This result originates from the fact that the superselection structure of the physical Hilbert space and the relation between fields and observables is governed by a compact gauge group G [24]. We mention as an aside that the in a sense opposite problem of whether one can derive from the partition functions relevant data of the superselection sectors, such as their statistical dimensions d_σ (Kac-Wakimoto formulas [36]), has recently received attention in the context of low dimensional theories, cf. for example the discussion by B. Schroer [37]. Unfortunately, our estimates are too weak to make any general statements on this problem. But Proposition 4.2 seems to support the idea that the statistical dimensions d_σ enter in a universal way in the partition functions of the respective canonical ensembles.

Let us mention in conclusion two interesting problems.

(i) It seems plausible that Conditions P and C, respectively N_p, are not completely independent. It would be of great interest to *derive* information on the mass spectrum, similar to Condition P, from suitable compactness or nuclearity conditions involving the observables. A related question is: do there exist theories where Conditions C or N_p are satisfied, but the maps $\Theta_{\beta,\mathcal{O}}$ of the field algebras into the physical Hilbert space are not compact or nuclear?

(ii) Another interesting issue is the formulation of compactness and nuclearity conditions in terms of the modular operators of the theory, which are affiliated with the vacuum and the local algebras [3]. This approach has the advantage that it can also be applied to generally covariant quantum field theories where time-translations are not a global spacetime symmetry and the formulation of compactness conditions in terms of H is no longer possible. It would therefore be desirable to exhibit physically significant conditions in terms of the modular operators which allow one to derive nuclearity properties of field algebras from corresponding properties of the observables. A relevant step in this direction is the computation of modular operators given in [38]. But one would need more specific informations on the

spectral properties of the relative modular operators appearing in these formulas for the derivation of such a result.

34

References

[1] D. BUCHHOLZ, E.H. WICHMANN, "Causal independence and the energy–level density of states in quantum field theory", *Commun. Math. Phys.* **106**, 321–344 (1986).

[2] D. BUCHHOLZ, C. D'ANTONI, K. FREDENHAGEN, "The universal structure of local algebras", *Commun. Math. Phys.* **111**, 123–135 (1987).

[3] D. BUCHHOLZ, C. D'ANTONI, R. LONGO, "Nuclear maps and modular structures II: Application to Quantum Field Theory", *Commun. Math. Phys.* **129**, 115–138 (1990).

[4] H.J. BORCHERS, R. SCHUMANN, "A Nuclearity condition for charged states", *Lett. Math. Phys.* **23**, 63–77 (1991).

[5] D. BUCHHOLZ, J. YNGVASON, "Generalized nuclearity conditions and the split property in Quantum Field Theory", *Lett. Math. Phys.* **23**, 159–167 (1991).

[6] D. BUCHHOLZ, P. JUNGLAS, "On the existence of equilibrium states in local quantum field theory", *Commun. Math. Phys.* **121**, 255–270 (1989).

[7] H. NARNHOFER, "Entropy density for relativistic quantum field theory" in *The state of matter*, eds. M. Aizenman, H. Araki, Singapore, World Scientific 1994.

[8] J. BROS, D. BUCHHOLZ, "Towards a relativistic KMS condition", *Nucl. Phys. B* (to appear).

[9] R. HAAG, J. SWIECA, "When does a quantum field describe particles?", *Commun. Math. Phys.* **1**, 308–320 (1965).

[10] V. ENSS, "Characterisation of particles by means of local observables", *Commun. Math. Phys.* **45**, 35–52 (1975).

[11] D. BUCHHOLZ, "On particles, infraparticles, and the problem of asymptotic completeness" in *VIIIth International Congress on Mathematical Physics*, Marseille 1986. Singapore, World Scientific 1987.

[12] D. BUCHHOLZ, M. PORRMANN, U. STEIN, "Dirac versus Wigner: Towards a universal particle concept in local quantum field theory", *Phys. Lett.* **B 267**, 377–381 (1991) .

[13] A. EINSTEIN, *The meaning of relativity* (Appendix 2), Princeton N.J. Princeton University Press 1955.

[14] R. HAAG, *Local Quantum Physics*, Berlin, Heidelberg, New York, Springer 1992.

[15] D. BUCHHOLZ, P. JACOBI, "On the nuclearity condition for massless fields", *Lett. Math. Phys.* **13**, 313–323 (1987).

[16] K. FREDENHAGEN, P. HERTEL, unpublished manuscript 1979.

[17] D. BUCHHOLZ, M. PORRMANN,"How small is the phase space in quantum field theory?", *Ann. Inst. H. Poincaré* **52**, 237–257 (1990) .

[18] S. DOPLICHER, R. LONGO,"Standard and split inclusions of von Neumann algebras", *Invent. Mat.* **75**, 493–536 (1984).

[19] S. DOPLICHER, R. LONGO,"Local aspects of superselection rules II", *Commun. Math. Phys.* **88**, 399–409 (1983).

[20] D. BUCHHOLZ, S. DOPLICHER, R. LONGO, "On Noether's theorem in quantum field theory", *Ann. Phys.* **170**, 1–17 (1986).

[21] S. DOPLICHER, R. HAAG, J.E. ROBERTS,"Local observables and particle statistics," I *Commun. Math. Phys.* **23**, 199–230 (1971), II *Commun Math. Phys.* **35**, 49–95 (1974).

[22] A. PIETSCH, *Nuclear locally convex spaces*, Berlin, Heidelberg, New York, Springer 1972.

[23] H. JARCHOW, *Locally convex spaces*, Stuttgart, Teubner 1981.

[24] S. DOPLICHER, J.E. ROBERTS,"Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics", *Commun. Math. Phys.* **131**, 51–107 (1990).

[25] J.E. ROBERTS, "Spontaneously broken gauge symmetries and superselection rules" in *Procedings of the International School of Mathematical Physics*, Camerino 1974, ed. G. Gallavotti, Università di Camerino 1976.

[26] D.BUCHHOLZ, S. DOPLICHER, R. LONGO, J.E. ROBERTS, "A new look at Goldstone's theorem", *Rev. Math. Phys.* Special issue, 47–82 (1992).

[27] J.J. BISOGNANO, E.H. WICHMANN,"On the duality condition for a hermitean scalar field", *J. Math. Phys.* **16**, 985–1007 (1975). "On the duality condition for quantum fields" *J. Math. Phys.* **17**, 303–321 (1976).

[28] J. YNGVASON,"A note on essential duality", *Lett. Math. Phys.* **31**, 127–141 (1994).

[29] D. GUIDO, R. LONGO,"Relativistic invariance and charge conjugation in quantum field theory", *Commun. Math. Phys.* **148**, 521–551 (1992).

[30] D. BUCHHOLZ, S. DOPLICHER, R. LONGO, J.E. ROBERTS, "Extensions of automorphisms and gauge symmetries", *Commun. Math. Phys.* **155**, 123–134 (1993).

[31] D. BUCHHOLZ, K. FREDENHAGEN,"Locality and structure of particle states", *Commun. Math. Phys.* **84**, 1–54 (1982).

[32] E. HEWITT, K. ROSS, *Abstract harmonic analysis* II, Berlin, Heidelberg, New York, Springer 1970 .

[33] C. D'ANTONI, S. DOPLICHER, K. FREDENHAGEN, R. LONGO,"Convergence of local charges and continuity properties of W^*–inclusions", *Commun. Math. Phys.* **110**, 325–348 (1987).

[34] O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics* II, Berlin, Heidelberg, New York, Springer 1981.

[35] H. Araki, R. Haag, D. Kastler, M. Takesaki, "Extension of KMS states and chemical potential", *Commun. Math. Phys.* **53**, 97–134 (1977)

[36] V.G. Kac, M. Wakimoto,"Modular and conformal invariance constraints in representation theory of affine algebras", *Adv. Math.* **70**, 156–236 (1988).

[37] B. Schroer, *Some useful properties of rotational Gibbs states in chiral conformal quantum field theory*, preprint 1994 .

[38] T. Isola, "Modular structure of the crossed product by a compact group dual", *J. Oper. Theory* (to appear).